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Ulrich Felgner

Philosophy of Mathematics in Antiquity and in Modern Times

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Philosophy of Mathematics in Antiquity and in Modern Times

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Preface

This is a book written by a mathematician on selected topics from the philosophy of mathematics. These topics are all within the scope of a single question, which, although central to mathematics, is not a mathematical question in itself, but a philosophical one. It is the *question of the sources from which we draw when we prove mathematical theorems.* This question is connected with the question of the mode of existence of mathematical objects. On the basis of the texts that have been handed down to us, I would like to discuss how these and related questions have been discussed and answered from antiquity to the present time. In particular, I would like to present, carefully and critically, the answers given by PLATO, ARISTOTLE, EUCLID, DESCARTES, LOCKE, LEIBNIZ, HUME, TSCHIRNHAUS, KANT, DEDEKIND, FREGE, HILBERT and others. In doing so, I will also deal with the various points of view referred to as *constructivism, empiricism, finitism, formalism, logicism, platonism, psychologism, rationalism, structuralism*, etc. I have selected only those contributions by mathematicians and philosophers who have made serious efforts to find comprehensive answers to the questions posed above.

Even if much of what is presented here will be well known to the experts, I think that some connections will be revealed here that have hitherto been noticed little or not at all, and that explanations will be given that are new and that lead to a deeper insight into mathematicians' way of thinking in the different epochs.

I first gave a lecture on *Philosophy of Mathematics* in Tübingen in the winter semester 1980/1981, and from then on, I have given it at larger intervals, most recently in the winter semester 2011/2012 at the *Forum Scientiarum* in Tübingen. After that, I worked long and hard on my notes to be able to bring them into the present form.

The book was originally written in German and appeared in print in September 2020, under the title *'Philosophie der Mathematik in der Antike und in der Neuzeit'.* The word *'Antike'* (Antiquity) refers to the time when the Greek-Roman culture florished. The word *'Neuzeit'* refers to the time from the end of the Middle Ages onwards, i.e from ca 1450/1500 onwards. Unfortunately the term *'Neuzeit'* cannot be translated into English. The dictionaries suggest the translation *'modern times'*, but this is not appropriate. This is because one always speaks of *'modern times'* when traditions, which had been previously valid and of fundamental importance, are broken. Such *'modern times'* occurred at the end of antiquity around the middle of the 5th century, provoked by the fall of Rome and the transition from the pagan religions to Christianity. But another period of *'modern times'* also started at the end of the Middle Ages and then a further one in the decades around 1900. This all means that the term *'Neuzeit'* is not correctly translated by the term *'modern times'*. - The only thing I can do in this situation is to emphasize that in the title of this book the term *'modern times'* will be used in the sense of the period from 15th to the end of 20th century.

The present book is a translation of the above mentioned book, however revised and considerably expanded. Various chapters have been completely rewritten.

The present book also contains a lot of citations from works of various authors. Most of these citations are given in the original language and in translations into English. It is necessary to add to the translated texts the original wording. Notice, that with translations from one language into another, one unfortunately encounters problems similar to the one mentioned above, again and again. Misinterpretations can only be avoided if one specifies the text in the original languages in addition to the translations. We want to take this into account in this book as often as necessary. All French, German, Greek, Italian, Latin and Spanish texts and phrases are translated by the author unless otherwise stated.

I started writing this book quite some time ago. During that time I have been helped by so many people that I cannot name them all here. Especially the discussions with students, doctoral candidates and colleagues after my lectures on Philosophy of Mathematics were very helpful. I would like to thank them all very much. But I would also like to thank my colleagues with whom I exchanged letters, who read some parts of my prepared notes, and who asked me critical questions and encouraged me to make some corrections, especially Mrs. LAURA CARRARA, Mrs. ANKE THYEN, Mr. MARTIN WAD THORSEN and Mr. WILFRIED SIEG, all of whom made themselves available to me as discussion partners. Special thanks go to Mr. WALTER PURKERT, and also to the unnamed reviewer of the original German version, who read the whole manuscript carefully, pointed out errors to me and suggested corrections and also additions.

I would like to thank my wife Cornelia very much for her support and assistance in all phases of the creation of this book.

Finally, I would like to express my thanks to Birkhäuser Verlag for their kindness and help in the production of the book.

Tübingen Ulrich Felgner
December 6, 2022

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Introduction

"Les Mathématiciens ont autant besoin d'estre philosophes que les philosophes d'estre Mathématiciens."

GOTTFRIED WILHELM LEIBNIZ in a letter to NICOLAS MALEBRANCHE dated 13/23 March 1699.

The topic of this book is the *Philosophy of Mathematics.* Philosophy of Mathematics is understood as an effort to clarify such problems and questions that mathematics itself raises, but cannot solve or answer with its own methods.

What is the *philosophical* aspect of this effort? *Philía tou sophou* (Φιλία τοῦ σοφοῦ) is, in Greek, *the love of understanding, insight, knowledge*, and it is from this that the word 'philosophy' is derived. The philosophy of mathematics will thus be about a committed, serious (loving) effort to understand the problem under consideration for its own sake, in order to finally arrive, after a critical examination, at a conviction that one can represent and defend.

What are the most distinguished questions that have been asked again and again in the philosophy of mathematics and that are still controversial today, or at least have not yet been fully answered? Most likely, these continue to be the questions about the *ontological status* of mathematical objects and the *epistemological status* of mathematical theorems.

The word *ontology* is used to describe the study of what 'is' (what exists) and in what way it 'is'. In Greek, *to on* (τὸ ὄν) is *the being*, which is derived from the verb *einai* (εἶναι): '*to be*', '*to exist*'. Thus, in ontology, the *mode of being* of objects is discussed.

Status is a Latin word meaning something like 'the state', the condition, the position, the situation. The 'ontological status' of an object is therefore the position of the object in relation to its mode of being, that is, what can be said about the state of its being.

In Greek, *epistêmê* (ἐπιστήμη) means 'understanding', 'knowledge', and thus *epistemology* is the theory of science.

In mathematics itself, it is not common to ask what the nature of mathematical objects might be. However, we want to ask this question, and ask, for example, what numbers are, and in what sense they exist. Are they objects that have an existence, or are they just linguistic entities, i.e., names that do not actually name anything? In PLATO, one can read:

> "Stranger: *All number is to be reckoned among things which are? –*
> Theaitetos: *Yes, surely number, if anything, has a real existence.*"
> PLATO: *'Sophistes',* 238a-b.

and in HANS HAHN, the opposite:

> *"And because we do not need numbers as separate entities* [*"als eigene Wesenheiten"*], *... we do not want to accept such entities."*
> HANS HAHN: *'Überflüssige Wesenheiten (Occams Rasiermesser)'*,1930 (reprint 1988,p. 34).

With ARISTOTLE, it's much more subtle:

> *"So what we have to deal with is not whether the numbers exist, but in which way they exist*." ARISTOTLE: *'Metaphysics'*, Book XIII,1, 1076a36.

There does not even seem to be agreement on the question as to whether numbers exist as objects (individuals, things or entities).

But doesn't every mathematical theory concern objects of a certain kind for the purpose of uncovering facts that exist between these objects? Arithmetic, for example, deals with integers and seeks to find the laws that are valid within the realm of these numbers. Geometry is about points, lines, triangles, circles, angles, etc., and seeks to determine which laws apply here. So, for every mathematical theory, there are two basic questions:

In what sense do the objects of the various mathematical theories exist and from what sources do we draw when we prove propositions (theorems)?

The first question concerns *ontology* and the second question *epistemology.*

ARISTOTLE demanded, in the 6th book of his *'Metaphysics'* (1025b8-9), that, in every science, the realm of objects must be indicated first, i.e., the realm of things with which science should deal. Thus, in the case of mathematics, the question arises right at the beginning as to which things it should deal with. Do objects of pure mathematics exist? Where is the space that constitutes their reality? Are they objects of the real, natural world or of an ideal world? Are they entities that we have constructed in our minds, or are they merely fictions that exist within the various mathematical theories, just as, for example, Cinderella only exists in fairy tales? Where do they get their being from? Or do these objects not exist at all? Are they only linguistic entities? Is mathematics just *a game in a vacuum ("ein Spiel im luftleeren Raum"),* as THOMAS MANN described it somewhat sarcastically in his novel *'Königliche Hoheit'* (Berlin, 1909)?

What "things" are actually being discussed in mathematics, since one can't see mathematical objects at all, indeed (as a rule), one can't even imagine them vividly, and, moreover, one can't even know whether they actually exist somewhere? Do they belong to a realm of spirits or a realm of the dead, about which one can only talk in a wonderful, mysterious language? Do they belong to another world, and is the process of recognizing mathematical facts only a remembrance (*anamnêsis,* ἀνάμνησις) of the soul of prenatal knowledge, as taught by the followers of Orpheus (οἱ Ὀρφικοί), the Pythagoreans and PLATO? Or do all of the mathematical objects mentioned not exist there either, and is the manipulation of the mathematicians, with all of their beautiful squiggled signs, just one single *abracadabra* without reference to any realities?

Whether mathematical objects exist at all seems very doubtful, because nobody can say where they exist. But, nevertheless, mathematical objects exist for everyone, and they are *accessible to all mathematicians in all nations and at all times ("sie sind allen Mathematikern in allen Völkern und in allen Zeiten zugänglich"),* as EDMUND HUSSERL

once wrote.[1] The objects of mathematics seem to have an ideal objectivity, but it is still rather unclear what this means. None of these objects are in any way tangible.

Behind all these questions is the fact that mathematics does not obtain most of its concepts and objects from the real world through abstraction, *but gives them to itself* by using calculi and axiom systems to merely establish the formal laws and rules of how these things may be dealt with. Only formalisms are designed, but the objects about which the formalisms claim to speak are not presented. - Once again: Where is the space in which they actually exist?

Geometry and the arithmetic of natural numbers, as they were constructed by the mathematicians of antiquity, are still directly anchored in the real world. But it is still not at all clear whether all these abstract or idealized objects really exist somewhere and in what sense.

This problem becomes even clearer when we look at the higher objects of mathematics.

At the end of the Middle Ages, the idea of *negative numbers* slowly developed (LEONARDO OF PISA, NICOLAS CHUQUET, MICHAEL STIFEL and others), as did, in the early 16th century, the idea of *imaginary numbers* (GERONIMO CARDANO, RAFAEL BOMBELLI). At the end of the 17th century, the *infinitesimal quantities* (BONAVENTURA CAVALIERI, GOTTFRIED WILHELM LEIBNIZ, ISAAC NEWTON and others) and, in the 19th century, the *ideal numbers* of algebraic number theory (ERNST EDUARD KUMMER, RICHARD DEDEKIND) and the *transfinite Alephs* (GEORG CANTOR) were added, etc. All these numbers could no longer be obtained from the natural real world by abstraction; they could only be introduced with the help of calculi and axiom systems. None of them has a direct relation to the objects of the real world. - Where do all these mathematical objects get their existence from? Do they *really* exist, and what is their *nature*, or do they only have the shadowy existence that they owe to a calculus?

Let us take a look at what famous mathematicians in earlier centuries had to say about this. As prominent examples, I have selected statements about the imaginary unit $\sqrt{-1}$, about infinitesimal quantities and about the concept of a geometric point.

RAFAEL BOMBELLI (1526-1573) could explain how to use the sign $\sqrt{-1}$ in calculations, but he could not explain the *nature* of this number. GERONIMO CARDANO (1501-1576) wrote (in his *'Ars magna Arithmetica',* problem 38) that $\sqrt{-1}$ could be neither +1 nor -1, but some *third thing of a hidden nature* (*"quaedam tertia natura abscondita"*). GOTTFRIED WILHELM LEIBNIZ (1648-1716) wrote, in 1702, about the imaginary unity (cf. LEIBNIZ, '*Werke'* (GERHARD, Editor) Volume 5, p. 357):

> *"Itaque elegans et mirabile effugium reperit in illo Analyseas miraculo, idealis mundi monstro, pene inter Ens et non-Ens Amphibio, quod radicem imaginariam appellamus ".*
>
> [The divine spirit has found a fine and wonderful evasion in that miracle of analysis, the monster of the ideal world, almost an amphibium between being and not-being, which we call imaginary unity.]

---

[1] EDMUND HUSSERL: *'Die Frage nach dem Ursprung der Geometrie als intentional-historisches Problem'*, Supplement III of the work *'Die Philosophie in der Krisis der europäischen Menschheit'*. A separate, posthumous publication of this supplement was arranged by E. FINK *in* 1939; a reprint appeared in volume VI of the Husserliana, 1962, where the quotation is to be found on page 368.

In his *'Réflexions sur la Métaphysique du calcul Infinitésimal'* (Paris 1797), LAZARE NICOLAS MARGUERITE CARNOT (1753-1823) described the imaginary numbers as *"hiéroglyphes de quantités absurdes*" (op. cit. p. 53). Even for JACOB STEINER (1796-1863), imaginary numbers were *ghosts* ("*Gespenster*") that belonged to a *shadowy kingdom of geometry* (einem *"Schattenreich der Geometrie"*), as FELIX KLEIN reported in his '*Vorlesungen über die Entwicklung der Mathematik im 19. Jahrhundert',* Part 1, p. 130. The Irish philosopher and theologian GEORGE BERKELEY (1685-1753) once expressed himself very similarly, and called the infinitesimal quantities of analysis *"ghosts of departed quantities"* (*'The Analyst',* §35). - Can't even the mathematicians themselves clearly express what they are dealing with?

OSKAR PERRON, in his book on *'Nichteuklidische Elementargeometrie der Ebene'* (Teubner-Verlag, Stuttgart 1962, p. 11), gave the following definition of the concept of a point: *A point is exactly that what the intelligent, but harmless, uninformed reader imagines it to be* (*"Ein Punkt ist genau das, was der intelligente, aber harmlose, unverbildete Leser sich darunter vorstellt"*). Must one be satisfied with such ridiculous excuses? Is it really so difficult, if not impossible, even in geometry, to clearly define the basic concept of a point?

Even the question of what, for example, natural numbers are and whether or how they exist was still unanswered a good hundred years ago. The mathematician GOTTLOB FREGE wrote about this in 1899:

> *"Es ist doch eigentlich ein Skandal, daß die Wissenschaft noch über das Wesen der Zahl im unklaren ist. Daß man noch keine allgemein anerkannte Definition der Zahl hat, möchte noch angehen, wenn man wenigstens in der Sache übereinstimmte. Aber selbst darüber, ob die Zahl eine Gruppe von Dingen oder eine mit Kreide auf einer schwarzen Tafel von Menschenhand verzeichnete Figur sei, ob sie etwas Seelisches, über dessen Entstehung die Psychologie Auskunft geben müsse, oder ob sie ein logisches Gebilde sei, ob sie geschaffen sei und vergehen könne, oder ob sie ewig sei, selbst darüber hat die Wissenschaft noch nichts entschieden. Ist das nicht ein Skandal? Ob ihre Lehrsätze von jenen aus kohlensaurem Kalke bestehenden Gebilden oder von unsinnlichen Gegenständen handeln, weiß die Arithmetik nicht. (...) Die Wissenschaft weiß also nicht, welchen Gedankeninhalt sie mit ihren Lehrsätzen verbindet; sie weiß nicht, womit sie sich beschäftigt. (...) Ist das nicht ein Skandal?"*
>
> [It is actually a scandal that science is still unclear about the nature of numbers. The fact that there is still no generally accepted definition of the number would be acceptable, if one at least agreed on the matter. But even whether the number is a group of things or a figure written in chalk on a blackboard by a human hand, whether it is something spiritual, the origin of which psychology must tell us, or whether it is a logical entity, whether it is created and can pass away, or whether it is eternal, science has not yet decided anything about that. Is that not a scandal? Arithmetic does not know whether its theorems are about those things made of carbonated lime or about non-sensory objects ... Science does not know, therefore, what thought-content it associates with its theorems; it does not know what it is dealing with. (...) Isn't that a scandal?]
>
> G. FREGE in the foreword to his book *'Über die Zahlen des Herrn H. Schubert',* op. cit.

FREGE himself believed that he had found a solution to the problem. For him, the number *zero* was the class of all unfulfillable concepts, the number *one* the class of all concepts,

under each of which exactly one object falls, and the number *two* the class of all concepts, under each of which one and also another one but no further object falls, etc. FREGE therefore understood the natural numbers as *logical entities.* (He considered class-formation as a logical process.) With an unprecedented thoroughness, FREGE proved the fundamental theorems of arithmetic. But in the summer of 1902, BERTRAND RUSSELL noticed a fundamental error in FREGE*'s* theory that caused the whole edifice of ideas to collapse (cf. Chapter 14). Hence, even FREGE had not succeeded in answering the question as to what numbers actually are, where they exist and in what sense they exist.

We have to state, somewhat disillusioned, that until the end of the 19th century, all attempts to define what numbers actually are and what their ontological status is were unsuccessful. Just as unsuccessful were all attempts to define points, lines and surfaces, along with what all the other objects of mathematics are, where they exist and in what sense they exist. It is obvious that, since ancient times, mathematicians have failed in answering such questions satisfactorily.

This probably indicates that there are no correct answers to these questions at all, and that there is no realm of mathematical objects either outside of us or within us (in our soul, spirit, or mind) in any way. *We only have the formalisms, the systems of rules and axiom systems that describe how to deal with the signs that are supposed to designate the objects in question.*

In principle, this is also sufficient to be able to do mathematics. But no one is willing to take the *nominalist viewpoint* that is required. *For many reasons, it is desirable to have objects that are designated by the signs of formal systems, so that one does not have to restrict all mathematical work to dealing with series of signs of formal systems (i.e., the syntax of formal systems), but can rather deal with mathematical objects in one's thoughts.*

Can such a wish be fulfilled? What price must one pay to fulfil such a wish? There are questions upon questions. Behind all of these questions is the one big question:

- What is the *ontological status* of mathematical objects?

This book also deals with the question of how we gain mathematical knowledge, how we get to know something about mathematical objects and their relations to each other. Can a proof be based on sense-perception - or on *"reine Anschauung"* ('pure intuition') in the sense of KANT - or only on laborious step-by-step logical reasoning and conceptual argumentation? How do we prove the existence of a mathematical object? Must one be able to call it by its name? Or must one have a mental construction for it? Or is it sufficient to lead with the assumption of non-existence to a contradiction? Obviously, the answer depends on how one understands the manner of existence of mathematical objects and what means are available to gain knowledge.

- So, the second question is: what is the *epistemological status* of mathematical theorems?

Are mathematical propositions apodictic truths, as is often claimed even today, or do they have anything to do with truth at all? Is mathematics even remotely a *science* (*epistêmê* (ἐπιστήμη), scientia) that examines a given world of objects, or is it not rather an *art* (*technê* (τέχνη), ars) that deals with the development of formalisms and calculi?

I repeat the problems I have just mentioned:

In what sense do the objects of different mathematical theories exist and from what sources may we draw when we prove propositions (theorems)?

Since antiquity, a host of mathematicians and philosophers have commented on the above questions. However, their statements have often differed considerably. Many were content with a frivolously dashed off answer, with only a few actually trying to test their answer to see whether it could explain the phenomenon of mathematical thinking and whether it could serve as a basis for mathematics.

In this book, I would like to report on some answers that have been given in the course of history. The critical examination of the answers given by PLATO, ARISTOTLE, AURELIUS AUGUSTINUS, RENÉ DESCARTES, GOTTFRIED WILHELM LEIBNIZ, JOHN LOCKE, IMMANUEL KANT, BERNARD BOLZANO, RICHARD DEDEKIND, GOTTLOB FREGE, BERTRAND RUSSELL, LUDWIG WITTGENSTEIN, DAVID HILBERT, KURT GÖDEL and others will make us sensitive to the problems at hand. This should eventually lead us closer to the possible answers to the questions asked and to a deeper understanding of the way mathematics is built up on the basis of set theory, as is usually done today.

Although the problem of the ontological status of mathematical objects was posed as early as antiquity, it is still relevant today, in particular, the question of the mode of existence of finite and infinite sets, the truth of the axiom of choice, the existence of 'inaccessible' uncountable cardinal numbers, etc. For the creators of set theory (BERNARD BOLZANO, GEORG CANTOR, ERNST ZERMELO and others), an *extension of a concept* (*"étendue de l'idée", "Begriffsumfang"*) was always a set, i.e., also an 'individual' that has an existence in the mind (of a human being or of an omniscient god) or in an ideal world. This understanding of the concept of sets led to the conquest of the realm of infinite sets, which DAVID HILBERT had compared to a paradise, and which led to today's structuralist view of mathematics. However, this understanding of the concept of sets also led to the well-known antinomies of set theory (BURALI-FORTI 1897, ZERMELO-RUSSELL 1900/1901, etc.) and to the *foundational debate* at the beginning of the 20th century. We must find a way out of this dilemma.

Many other questions are connected with the question of the manner in which mathematical objects exist. For example, one may ask whether it is at all useful in mathematics to work with mathematical objects, which, in some sense, have an existence. Would life be easier if all metaphysics were banned from mathematics and pure nominalism were advocated? - It is certainly useful to investigate the consequences of the *nominalist standpoint* in mathematics. It is equally useful to explore the benefits of ontological assumptions. Should it be a goal in mathematics to manage to live upon as few ontological assumptions as possible (*parsimonia ontologiae*) or is it not more sensible to provide a world of mathematical things that is as rich as possible (*abundantia ontologiae*)? What is the price we have to pay when we make ontological assumptions, and what do we gain by making them? - This can only be explored using mathematical logic as it was developed in the 20th century.

In the 20th century, the term ‘mathematical theory’ was developed. In such a theory, everything is laid down that is necessary to build up a mathematical discipline. Everything that is relevant from the point of view of mathematics about the nature of mathematical objects, their mode of being and their epistemological status for the practice of mathematics is explicitly laid down here. Thus, within such theories, one can do mathematics without coming into contact with philosophical problems. However, if one wants to justify why the respective theories are significant and may claim universal acceptance, and why the underlying logic of the first (or even the second) order is adequate and why the theories are ‘true’ or at least ‘correct’ (i.e., free of contradictions), and which superordinate (extra-mathematical) problems can be treated in these theories, then philosophical reflections are still inevitable.

Of the topics we want to deal with in this book, many have now been laid out and we can proceed to the development.

Part I
Philosophy of Mathematics in Antiquity

We discuss three drafts for an *ontology* and *epistemology* of mathematics.

The oldest draft is by PLATO: The mathematical objects belong to the world of ideas. The soul and intellect succeed in recognizing the fundamental mathematical truths by remembering their prenatal participation in the world of ideas. We report on this in Chapter 2.

A second draft is by ARISTOTLE: the basic mathematical concepts are bound to the objects of the sense-perceivable world. In mathematics, therefore, one studies the objects of that world, but one studies them only in terms of their numbers or their geometric formes, and so on. We report on this in Chapter 3.

A third draft was worked out by the mathematicians themselves. It is the *'axiomatic treatment'* of a mathematical theory. In Chapter 4 we discuss the axiomatic method as it is found in the '*Elements*' of EUCLID.

Before that we want to report in Chapter 1 about the genesis of mathematics in ancient Greece. With this we want to emphasize some characteristic features of mathematics and make clear why mathematics poses ontological and epistemological problems.

In the Chapters 5 and 6 we discuss the problems that arise in dealing with the infinite. In Chapter 5 we discuss the *finitist position* of the Greek mathematicians and in Chapter 6 we report on the *paradoxes of infinity* as they were established by ZENO and then discussed by ARISTOTLE and the philosophers in the Middle Ages.

Chapter 1
The Concept of Mathematics

In this introductory chapter, we want to discuss one of the earliest mathematical discoveries, namely, the discovery of the existence of *incommensurable quantities* by the Pythagoreans about two and a half thousand years ago.

With this discovery, something new was created, which the Greeks called *"mathematics"*. We will examine what this novelty consisted of compared to the older Egyptian and Babylonian arithmetic and geometry. We will also look at the original, colloquial meaning of the word *"mathematics"*. In particular, we will make it clear that this novel mathematics poses ontological and epistemological problems that did not exist before.

1.1 The discovery of incommensurable quantities

So, let us start our investigations with a discussion of the discovery of incommensurable quantities. This discovery comes from the early days of mathematics, around 470/450 B.C.E. It was a time in which the Greek culture was experiencing an extremely rich blossoming. Shortly before that, in the battles of Marathon (490 B.C.E.), Sálamis, Himera (480 B.C.E.) and Plataeae (479 B.C.E.), the hostile armies of the Persians and Carthaginians had been defeated, and this victory, which the Greeks attributed more to their virtue (*aretê*, ἀρετή) and their sacrificial courage for the community than to their military strength, made them proud and self-confident. Greece became a politically significant region. An economic boom began, causing the culture to flower. It was the time of the great tragedian poets

AESCHYLUS (525-455), SOPHOCLES (497-405), EURIPIDES (485-406)

and the (pre-Socratic) natural philosophers

PARMENIDES of Elea (515-445), EMPEDOCLES of Akragas (ca. 500-430),
ZENO of Elea (490-?), ANAXAGORAS of Klazomenai (ca. 498-427) and others.

At this time, a sensational discovery was made in the circle of Pythagorean mathematicians, a discovery that we today render in very abbreviated form as proof of the irrationality of $\sqrt{2}$. However, this is only an imperfect description of the discovery. Let us take a closer look at the result and its background.

The proof of the irrationality of $\sqrt{2}$ probably goes back to the mathematician HIPPASUS (Ἵππασος), who was born in Metapont, an Achaean colony on the Gulf of Taranto

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(southern Italy), about 50 km west of Taranto (cf. KURT V. FRITZ, op. cit. 1945/1965). HIPPASUS belonged to the Pythagorean community. PYTHAGÓRAS (Πυθαγόρας) of Samos (c. 570/560 - 480 B.C.E.) had founded this community in Kroton (southern Italy) in about 520 B.C.E. It was a religious-philosophical community, which was divided into an inner circle (the esotericists, ἐσωτερικοί) and an outer circle (the exotericists, ἐξωτερικοί).

PYTHAGORAS inaugurated the members of the inner circle into his more profound doctrines. He encouraged them to perform their own research, which would lead them to results and insights that would confirm, substantiate and expand the teachings he presented. They were thus privileged to learn the teachings by way of understanding. Therefore, they were also called "Mathêmatikoi" (μαθηματικοί).[1] HIPPASUS belonged to this inner circle of "mathematicians".

To the members of the outer circle, PYTHAGORAS only communicated the more easily comprehendible statements of his doctrine without further explanations. These members were only granted the pleasure of listening to the teachings. Therefore, they were also called "Akousmatikoi" (ἀκουσματικοί).

The research of the members of the inner circle was mainly concerned with the Orphic-Pythagorean doctrine of the harmonic structure of the world. This doctrine says that the world arose from the disordered, shapeless and attributeless primordial mass, the chaos (χάος), and that it was a God (or perhaps nature itself) who created the well-structured, harmoniously ordered world out of the chaos (cf. HESIODUS *'Theogony'*, PLATO *'Timaios'* 53b-c, OVID *'Metamorphoseon'* I,5-74, T. LUCRETIUS CARUS *'De Rerum Natura'* V,416-507, etc.). The research led the Pythagoreans to a deeper understanding that

> *"in this universe everything is harmoniously ordered by number and ratio."*
> IAMBLICHOS*: 'De vita Pythagorica liber'*, [XII] 58.

According to PYTHAGORAS and his disciples, the *harmonic* order was revealed in which the individual parts are expressible in simple ratios between natural numbers. In this respect, the Pythagoreans spoke of *everything being numbered*. The Pythagorean PHILOLAOS from Kroton (he lived around 400 B.C.E.) described this fundamental conviction in the following words:

> *"And indeed, everything you can see has a number. For without it nothing can be grasped or recognized".* (cf. H.DIELS - W.KRANZ: '*Die Fragmente der Vorsokratiker'*, I, p. 408).

With ARISTOTLE, we read, in his *'Metaphysics'*, Book A, 986a:

> *"And when they* (the Pythagoreans) *saw that the properties and ratios of musical harmonies are determined by numbers, and when it seemed to them that all other things are also reproduced in their whole nature according to numbers, and that numbers are the first in the whole universe, they thought that the elements of numbers are the elements of all things, and that the whole heaven is harmony and number."*

Thus, according to the Pythagoreans, natural numbers and their ratios are the most important keys for understanding the world, its structure and order.

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[1] The verb 'manthánein' (μανθάνειν) generally means 'to learn', namely, 'to learn by reflection', as opposed to 'to learn by repeated practice' and 'to learn by experience'. We will discuss this word in more detail below.

In the field of music, for example, they discovered that the melodious intervals, now called the octave, the fifth and the fourth, are determined by the simple numerical ratios 1:2, 2:3 and 3:4. These proportions indicate how the corresponding sections of the sounding string that are not kept silent relate to each other. The Pythagoreans extended the system of these harmonic intervals by the (Pythagorean) whole-tone 8:9 (the fifth is thus reached by a fourth and a whole-tone: $\frac{2}{3} = \frac{3}{4} \cdot \frac{8}{9}$) and the somewhat too large *"Pythagorean major third"* 64:81 (which consists of two whole-tone steps of equal size: $\frac{64}{81} = \frac{8}{9} \cdot \frac{8}{9}$). They did not include the much better sounding *"harmonic major third",* which is given by the ratio 4:5, in their system of tones, since they did not succeed in breaking this interval down to two whole-tone steps of equal size, which, in addition, are also describable in terms of ratios between natural numbers (such a whole-tone [2] would be determined by the mean proportional x between $\frac{4}{5}$ and 1, i.e., $\frac{4}{5}:x = x:1$).

Also, in their investigations in the fields of geometry and arithmetic, the Pythagoreans repeatedly came across distances that can only be described as mean proportionals. For example, the problem of squaring a rectangle with sides s and t amounts to solving the equation $s:x = x:t$. In particular, doubling the square with side s amounts to solving the equation $s:x = x:2s$. It is quite easy to solve this equation geometrically, i.e., to construct, for any given straight line s, another straight line of length x such that $s:x = x:2s$ (Hint: apply either the Theorem of THALES or the Pythagorean Theorem). However, it is a problem as to whether we can also express explicitly how much longer x is compared to s. This means, in the terminology of the Pythagoreans, whether the ratio $s:x$ is equal to a ratio between positive integers.

The Pythagoreans were finally able to prove that the answer usually comes out in the negative. It was probably HIPPASUS of Metapont who proved (around 470/450 B.C.E.) that not even the mean proportional x between the two distances of lengths 1 and 2 can be represented as a ratio of natural numbers. Somewhat reformulated, he proved that the side and diagonal of a square are not measured by a common measure, or, in other words:

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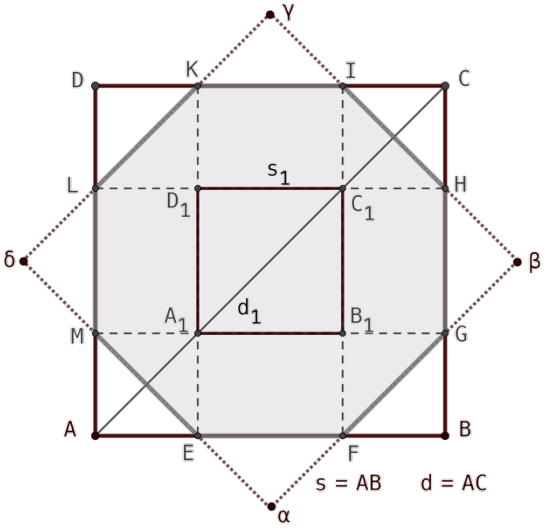
[2] The later history is quite interesting. RAMIS DE PAREJA proposed in 1482, to replace the *Pythagorean major third* 64:81 with the much better sounding *harmonic major third* 4:5. In 1571, ZARLINO also proposed replacing the *Pythagorean whole-tone* 8:9 with the arithmetic mean between the so-called *large whole-tone* 8:9 and the *small whole-tone* 9:10, which is $\frac{161}{180} = 0{,}894444\ldots$ Notice that this is almost the same as the geometric mean $\sqrt{\frac{8}{9} \cdot \frac{9}{10}} = \sqrt{\frac{4}{5}} = 0{,}894427\ldots$ which, however, was not considered here because of its irrational character. The new system was called *'meantone temperament'* ('Mitteltönige Stimmung'), because of the significant use of the arithmetic mean in the definition of a *'whole-tone'*. Thus, the problem with the whole-tone and the major third was more or less solved. But the resulting tone-system was still not yet altogether 'well-temperated', e.g., the octave is not the same as three major thirds, etc. After the introduction of various further slight alterations during the Baroque period (proposed by WERCKMEISTER, KIRNBERGER and many others), in the course of the nineteenth century, all these unequally tempered systems were finally given up and replaced with the modern *'equal temperament'* ('gleichstufige Temperatur') in which all intervals are powers of the semitone $\sqrt[12]{1/2}$. The result was an obliging system of tones, almost perfect, in which, however, only the octaves are pure and all other intervals are more or less impure.

Theorem: *The ratio of the lengths of the diagonal and the side of a square cannot be expressed with natural numbers; it is irrational* (ἄρρητος).

Notation. *AB* denotes the straight line with the end points *A* and *B*, and *AB* = *CD* means that the two straight lines *AB* and *CD* are of equal length (i.e., they are congruent).

Proof [3]. We consider a square with the vertices *A,B,C,D*, its side *s* = *AB* and its diagonal *d* = *AC* = *BD*. Add a second square of the same size, but rotated by 45º around its center, and let α, β, γ, δ be its vertices. Notice that the region in the interior of both squares forms an octagon with vertices E,F,G,H,I,K,L,M. By symmetry, it is a regular octagon, and hence all its sides are of equal length: EF = FG = GH = … = LM = ME.

Draw the straight lines EK, FI, GM and HL, and let the intersecting point of EK and GM be A_1. Similarly, let the intersecting point of FI and GM be B_1, the intersecting point of FI



[3] The only proof from Antiquity handed down to us is sketched in the *'Analytica priora'*, XXIII & XLIV, of ARISTOTLE and with full details in the *'Element's*, X §115a, of EUCLID. It is shown there that, from the assumption that the side and diagonal of a square are commensurable, it would follow that there are natural numbers that are both even and odd, which is a contradiction. It might be that this is the original proof as given by HIPPASUS. However, it is also possible that a purely geometric proof was given originally that was not handed down to us. Therefore, OTTO TOEPLITZ (1930) and KURT VON FRITZ (1945) tried to reconstruct such geometric proofs. Both reconstructions are quite convincing. The reconstruction presented here is taken from our paper *'Über Krisensituationen in der Mathematik und wie sie gelöst wurden'* (U. FELGNER, 2023, op. cit.). This reconstruction is related to that credited to TOEPLITZ, but is a bit simpler, since no knowledge of geometrical theorems is presupposed. However, all these reconstructions are based on the euclidean algorithm.

and HL be C_1 and the intersecting point of EK and HL be D_1. Let $s_1 = A_1B_1$ be the side and $d_1 = A_1C_1$ the diagonal of the interior square A_1,B_1,C_1,D_1. Notice that s_1 is nothing other than the length of a side of the regular octagon, $s_1 = EF = FG =...$.

First claim: $s_1 = d - s$. Proof: subtract αβ from AC.

Second claim: $d_1 = s - s_1$. Proof: subtract twice of the half of the diagonal of the small square AEA_1M from αβ.

Thus, s_1 and d_1 are obtained from s and d through a process known as the euclidean algorithm, also called *"antiphairesis"* (ἀνθυφαίρεσις).

Iteration: The process that led from the square A,B,C,D to the interior square A_1, B_1, C_1, D_1 may be repeated and will then lead from A_1, B_1, C_1, D_1 to a further, much smaller interior square A_2, B_2, C_2, D_2. This procedure can be iterated and will not stop after a finite number of steps. Hence, we obtain an infinite sequence of squares A_{n+1}, B_{n+1}, C_{n+1}, D_{n+1} with sides s_{n+1} and diagonals d_{n+1}, all of non-zero lengths. In analogy to the two claims made above, we will have:

$$s_{n+1} = d_n - s_n, \quad d_{n+1} = s_n - s_{n+1},.$$

Let us assume that s and d were commensurable, and let g be a common measure of them. Thus, s and d both are integral multiples of g. By the first and second claims, s_1 and d_1 are also non-zero integral multiples of g. By analogy, the same applies to s_n and d_n (for all integers n). This shows that $0 < g < s_n$ und $g < d_n$ for all integers n.

But $s = d_1 + s_1 > s_1 + s_1$, thus $s_1 < \frac{s}{2}$ and also, analogously, $s_{n+1} < \frac{s_n}{2}$. This shows that the sides s_n of the squares converge to zero, and hence, for some integers m, it holds that $0 < s_m < g$, a contradiction! Thus, we have to relinquish our assumption, and this means that, in a square, therefore, the length of the diagonals and the length of their sides cannot have any relationship that can be expressed as a ratio between positive integers: their relationship is *irrational* (inexpressible, ἄρρητος)! - Q.E.D.

This was a sensational discovery! The dogma of the Pythagoreans, *"that everything in this universe is ordered harmoniously according to number and ratio"* (see above), was thus refuted, which led to extremely violent arguments in the Pythagorean circle. However, for the further development of geometry (and mathematics as such), the result was of utmost importance.

If one imagines that (in the diagram used in the above proof) all strokes had to be drawn with ink or pencil on a sheet of paper - or, as was customary in antiquity, with a thin stick on a board sprinkled with fine sand, which was called an *abax* (ἄβαξ) or *abakion (*ἀβάκιον) - then one can only iterate the construction of the squares as long as one can still distinguish the points and lines from one another. The question of whether there is a common measure can obviously *not* be checked *empirically.*

In order to ensure that the process of *antiphairesis* is feasible and can be continued ad infinitum, the straight lines that occur must all exist, i.e., it must still be possible to

distinguish them all, and that means that the straight lines that occur *must all have no breadth!*

Never before in the history of geometry had it been necessary to ask the question of how broad geometric lines could be, but now this question arose for the first time, and the answer was that they should not have any breadth at all. In exactly the same way, the question arose as to how thick points can be, with the answer being that they must not have any extension.

But this also resulted in the consequence that the question as to whether the side and diagonal of a square are commensurable can obviously not be answered on the basis of sense perception. But, if one assumes that the lines all have *only length*, but *no breadth*, then one recognizes that the process of the antiphairesis described above can be continued *ad infinitum* and does not break off, and thus does not lead to a common measure! So, there is no common measure. Therefore, the diagonal and side line of a square are *incommensurable.*

On the basis of a geometry with extensionless points, breadthless straight lines, etc., there are thus quantities for which no common measure can be found at all. ARISTOTLE wrote about this in the 1st book of his *'Metaphysics'* (I,2, 983a15) and similarly in his *'Second Analytics'* (I, 71b27):

> *"It must seem wonderful to all those who have not yet grasped the reason why even the smallest size cannot be a common measure".*

The Pythagoreans also probably recognized that, in every regular pentagon, the diagonal and side are incommensurable. A (re?)-construction of such a proof was given by KURT VON FRITZ (1945/1965, op. cit.). [4]

What was decisively new in the argumentation of the Pythagoreans was that it could not refer to sense-perceptible points, lines, surfaces and bodies, but had to refer (in full generality) to idealized points (without extension), idealized lines (without breadth), and idealized surfaces (without thickness). Such idealized points, lines, surfaces and bodies had never been mentioned in the older Egyptian-Babylonian geometry, nor had it been in THALES (ca. 624-548/545 BCE). From then on, however, the Greek geometers spoke almost exclusively of these idealized objects. Additionally, EUCLID (ca. 340-270 B.C.E.) began the famous *'Elements'* (written around 300 b.o.c.) with the definitions:

Σημεῖόν ἐστιν, οὗ μέρος οὐθέν.	[A Point is that which has no parts.]
Γραμμὴ δὲ μῆκος ἀπλατές .	[A line is length without breadth.]
Ἐπιφάνεια δέ ἐστιν, ὃ μῆκος καὶ πλάτος μόνον ἔχει.	[A superficies is that which has only length and breadth.], etc.

A geometry was created that deals with things that one can neither see nor feel and that exist only for contemplation. It is not possible to determine empirically whether a geometric statement is true or false, but only to contemplate through the means of dialectics.

The older Egyptian and Babylonian arithmetic and geometry consisted largely of a

[4] See also WILBUR R. KNORR (1975), op. cit., and ÁRPÁD SZABO (1969), op. cit.

collection of methods for solving practical problems. The correctness of the proposed solution methods was observed in many (but not *'all'*) examples, and thus was assumed to be correct and generally valid by means of (incomplete) *induction*. The development of a *theoretical geometry* did not occur. The arithmetic and geometrical terms appeared only as adjectives (in colloquial language) and were used to speak about facts in the real world. Abstract and idealized objects subject to a *deductive* proof procedure were not formed in Egyptian-Babylonian arithmetic and geometry. This was reserved for Greek mathematics, and we have just witnessed the hour of birth of this new mathematics in the example discussed in detail [cf. O. NEUGEBAUER, op. cit., especially pages 120 and 122].

1.2 The concept of *'mathematics'*

The new *deductive* (!) arithmetic and geometry was soon labelled *"mathematics"*. We wonder what this word actually means (according to the original, colloquial sense of the word). What did the word mean before it became a generic term for arithmetic, algebra, geometry, etc.?

In colloquial Greek, educational subjects were often referred to as *mathêmata* (τὰ μαθήματα). The verb *'manthanein'* (μανθάνειν) means *"to learn"* in the sense of *"learning by (argumentative) instruction, learning by reflection"*, as opposed to *"learning by experience"*. The noun *mathêma*, or, in the plural, *mathêmata*, thus designates the object (or objects) of learning and teaching (cf. KURT VON FRITZ 1960, op. cit.). *Mathêsis* (μάθησις) is *learning by understanding*, the *acquisition of knowledge.*

BRUNO SNELL (op. cit., p. 72 ff.) points out that, as early as pre-Platonic times, *manthanein* (μανθάνειν) had the meaning of *making something spiritually one's own* (*"sich geistig etwas zu eigen machen"*). This verb should be distinguished from the verb *"to learn by practicing"* (*didaskesthai,* διδάσκεσθαι).

The Greek word *manthanein* has the Indo-Germanic root *"mendh-"* (= to be mentally aroused, to direct one's mind to something, to think). The same root gives us

- in English, the word *'mind',*
- in German, the word *'mahnen'* (i.e., to remind),
- in Latin, the word *'mens'* (i.e., the ability to think, the mind, the conscience, the admonishing inmost soul) and
- in Sanskrit, the word *'man'* (= to think, to mean) and the nouns *'manas'* (= the inner sense, mind) and *'mantra'* (= a word that, when recited, binds the mind to the contents of the word, and thus protects it from other thoughts).

The *mathêmata* were generally scientific subjects such as rhetoric, music, arithmetic, geometry, astronomy, optics, etc. This generic iteration of the word *'Mathêma'* was used again and again by ARISTOPHANES (e.g., *'Aves'* 380, *'Nubes'* 1231), EURIPIDES (*'Medea'* 1078), THUCYDIDES (II,39), PLATO *('Sophistés'* 219c, *'Tímáios'* 88b, *'Politeia'* 504-505) and many others. But the word lost its colloquial meaning in the course of time and was ultimately only used to designate the scientific subjects arithmetic and geometry.

As already indicated above, among the Pythagoreans, the 'Mathematikoi' were those who were offered the chance to learn the teachings of their master by way of understanding (thinking, checking, verifying, reflection). The most spectacular achievement of the

Mathematikoi was the statement about the existence of incommensurable quantities. In the following decades and centuries, this achievement led to a thorough transformation of *geometry* into a *geometry on an axiomatic basis* (EUCLID), and to the construction (in its very first steps) of a *real analysis* (EUDOXUS, ARCHIMEDES and others). All of this contributed to the result that, ultimately, only the subjects of arithmetic, algebra, analysis and geometry were called "mathematical subjects".

The word ‘mathematics’ has rarely been translated into other languages. *Cassiodorus* (ca. 490-583) translated it, in his *'Institutiones'*, as *doctrina.* He wrote in the second book of these *'Institutiones'*:

> *"Mathematica, quam Latine possumus dicere "doctrinalem", scientia est quae abstractam considerant quantitatem."*

In Latin, *docere* means "to teach, instruct, lecture". *A doctor* is "a teacher", a *doctrix* is "a female teacher" and *doctrina* is "the teaching, the instruction, the erudition and knowledge communicated by instruction". *Doctrina,* in particular, is a system of principles established by philosophy or by science, i.e., a self-contained teaching, a self-contained system of statements.

In a letter to GABRIEL WAGNER in 1696, GOTTFRIED WILHELM LEIBNIZ described mathematics as *Wiß-Kunst.* In Dutch, the term *Wiskunde* has been common since the 18th century. It is remarkable that LEIBNIZ considered mathematics as an *art* (τέχνη, ars) and not, like CASSIODOR, as a *science* (ἐπιστήμη, scientia).

It is also noteworthy that, in pre-Platonic times, geometry was occasionally called *Historia.* For example, IÁMBLICHOS states, in his book *'De vita Pythagorica liber'*, [XVIII],89, that PYTHAGORAS called geometry *Historia* (ἱστορία):

> ἐκαλεῖτο δὲ ἡ ἡ γεωμετρία πρός Πυθαγόρου ἱστορία.
> [PYTHAGORAS himself called geometry ‘Historia’.]

KURT VON FRITZ (1952 loc. cit.) wrote that the word ‘Historia’ is derived from the Indo-Germanic stem ‘vid’, which is found in the Latin word *videre*:

> A ἵστωρ is someone who has seen something, an eyewitness; ἱστορεῖν means either to look at something as an eyewitness and report about it, or, in a more derivative sense, to interview eyewitnesses and report what you have learned from them. "Historia" in its original sense, therefore, is ultimately based on eyewitnessing.
>
> (translated from K.V. FRITZ 1952, op. cit., p. 202.)

‘Historia’ therefore means, as B. SNELL (op. cit., p. 62) has put it: *Investigation by those who know from their own experience* (*“Nachforschen bei denen, die aus eigener Anschauung Bescheid wissen”*). ‘Historia’ is thus also an appropriate description of geometry as it was practiced in ancient Egypt and Babylon. In this ancient form of geometry, when it came to seeing the validity (or truth) of a geometric assertion, one was allowed to resort to sense-perception (laying geometric figures on top of each other to prove their congruence, pushing around and inserting given lengths, using mechanically drawn curves, etc.). It is remarkable that, in later Greek geometry, recourse to sense-perception was no longer permissible and proofs could only be made conceptually with rational arguments.

1.3 The occurrence of ontological problems

We have described the development that geometry has undergone, quite uncritically. From the Pythagoreans onwards, the field of objects with which geometry is concerned was no longer 'of this world', but rather a field of *imagined* or *fictional* unreal things. But this also created new problems. Is the unreal geometric *point without extension* a 'thing' at all, is it a real object, or is it perhaps merely a 'nothingness'? Are the *imagined* (or *fictional*) points, lines, circles, surfaces, angles, etc., real objects?

- In what sense can they be regarded as things (or individuals)? This, like those above, is a question that concerns ontology. They concern the way in which mathematical objects exist. Have the extensionless points, the breadthless lines, etc., been constructed at all? In what world do they exist? Do they exist independently of our thinking? Do they belong to the created world or are we the ones who create them? To whom do they owe their existence?
- MORITZ SCHLICK (1882-1936) wrote, in his *'Allgemeine Erkenntnislehre'* (Berlin 1918, p. 117): *Lines without breadth are not really conceivable* ("*Linien ohne Breite sind nicht wirklich vorstellbar"*). The question arises as to whether geometry still has something to do with reality. To what does it refer: sense-perceivable facts or rather a framework of concepts?
- In what way is it possible for us to recognize the relationships that exist among various mathematical objects? This is a question that concerns epistemology, i.e., the way in which we recognize mathematical facts.

Already in antiquity, all of these questions were being discussed in an atmosphere of tremendous controversy, with geometry being attacked, and even mocked, e.g., ARISTOPHANES in his play *'The Birds'*, CICERO in his *'Academiae Questiones'* (book II, liber Lucullus, XXXVI, 116-118), SENECA (in his 88th letter) and LUCIAN (ca.120-180 o.c.) in his dialogue *'Hermotios or Of the Philosophical Sects'*. LUCIAN, for example, wrote in that piece:

> (...) for even the highly admired geometry demands from the outset that one concedes to it obviously absurd conditions, since it wants to have admitted certain indivisible points and lines without breadth and the like nonsense, and in that it builds on such a worm-eaten foundation, but makes itself broad with demonstration and evidence.

Fifteen hundred years later, geometry was still being ridiculed for its extensionless points and infinitely thin lines. VOLTAIRE, in his satire '*L'Homme aux quarante écus*' (written in 1768), mocks the lines that have a length but no breadth (*"des lingnes qui ont de la longueur sans largeur"*) and calls all geometry a "charlatanry":

> *"Je fus très-content de l'aveu de ce sage mathématicien, et je me mis à rire, dans mon malheure, d'apprendre qu'il y avait de la charlatanerie jusque dans la science qu'on appelle la haute science."*
>
> [I was very pleased with the confession of this wise mathematician and began to laugh in my misfortune, because I had to learn that even in the so-called "high science", there is charlatanism]. VOLTAIRE*: 'Oeuvres Complètes',* Kehl 1785, Volume 45, pages 13-14.

Even if the mockeries of LUCIAN and VOLTAIRE are rather superficial, the problem remains as to

(1.) what the subject matter one speaks about in geometry is, and
(2.) from where one gets the insight into geometric truths.

This was the question we asked above:

In what sense do the objects of different mathematical theories exist and from what sources may we draw when we prove propositions (theorems)?

Is the foundation of geometry as wormy as LUCIAN thinks? Is the foundation that EUCLID laid in his *'Elements'* also '*worm-eaten*'? - We shall examine this in the following chapters.

The mathematicians themselves have hardly said anything about the nature of mathematical objects. Yet, the handling of mathematical objects is by no means without problems. If all considerations refer only to imaginary points, imaginary lines, etc., what role do mechanical aids (use of generalized compasses for constructing parabolas, insertion of a ruler for trisecting angles, etc.) play in geometry? Are curves that can *only* be generated mechanically permissible in geometry at all? Such curves do not even exist 'in thought'. The *quadratrix* of HIPPIAS of Elis (around 420 B.C.E.) is one such mechanically generated curve, allowing for angular trisection (even the division of any angle into n equal parts, $n \geq 2$ arbitrary) and the squaring of the circle. Is the Quadratrix permissible in geometric proofs of existence? Is the *kissoid* of DIOCLES (around 100 B.C.E.), with which the doubling of the cube can be performed, permissible in geometry? The same question arises for many other curves.

What do we mean when we say that the points without extension, the straight lines without breadth, the surfaces without thickness, etc., "would exist in the mind". What does that actually mean? Or do the geometric objects only exist in the axiomatically constructed geometries of EUCLID and HILBERT? In what sense are the points and straight lines, of which HIPPASUS spoke, objects of geometry? - What is the ontological status of geometric objects?

There are questions after questions. But they do not only concern geometric objects. Similar questions are equally relevant to "imaginary numbers" (R. BOMBELLI, L. EULER, C.F. GAUSS), "infinitesimal quantities" (B. CAVALIERI, G.W. LEIBNIZ, I. NEWTON), the "ideal numbers" of algebraic number theory (E.E. KUMMER, R. DEDEKIND), the infinite sets and the "Alephs" (G. CANTOR), etc. - We want to examine what answers have been found in the course of history. This will help us to find our own answer.

A number of philosophers, logicians and mathematicians have closely followed (and, in part, influenced) the development and redesign of mathematics and have presented their views in detail. To begin with, for the period from classical antiquity to the rebirth of antiquity in the 15th, 16th and 17th centuries, the following persons have to be named:

PLATO (ca. 427-347 B.C.E.), ARISTOTLE (384-322 B.C.E.), PLOTINUS (205-270), AURELIUS AUGUSTINUS (354-430), PROKLUS DIADOCHUS (ca. 411-485) and RENÉ DESCARTES (1596-1650),

and from the time of enlightenment,

> JOHN LOCKE (1632-1704), GOTTFRIED WILHELM LEIBNIZ (1646-1716), EHRENFRIED WALTER VON TSCHIRNHAUS (1651-1708), GEORGE BERKELEY (1685-1753) and IMMANUEL KANT (1724-1804).

In modern times, philosophers, logicians and mathematicians have worked together to address the fundamental problems raised by mathematics. These have included, above all,

> BERNARD BOLZANO (1781-1848), RICHARD DEDEKIND (1831-1916), GOTTLOB FREGE (1848-1925), EDMUND HUSSERL (1859-1938), DAVID HILBERT (1862-1943), BERTRAND RUSSELL (1872-1970), LUITZEN E.J. BROUWER (1881-1966), PAUL BERNAYS (1888-1977), LUDWIG WITTGENSTEIN (1889-1951) and KURT GÖDEL (1906-1978).

We will discuss their efforts in the following chapters.

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Chapter 2
PLATO's Philosophy of Mathematics

PLATO (Πλάτων) was born around 428 B.C.E. in Athens (or Aigina?). He descended from an old, highly respected Athenian family. At the age of about 20, he became acquainted with the then-sixty-two year old SÔCRATÊS (Σωκράτης). PLATO witnessed the trial through which SOCRATES was put (399 B.C.E.). He was deeply outraged and indignant about the injustice that had been visited upon SOCRATES. He subsequently left Athens and went to Megara to EUKLEIDÊS (Εὐκλείδης) and (according to tradition), some years later, to Cyrene, where he was introduced to mathematics by THEODÔRUS (Θεόδωρος). In the years 395 - 390 B.C.E., he was probably back in Athens, where he wrote the *'Apology'* and composed his first dialogues: *'Euthyphron', 'Gorgias', 'Ion', 'Kriton', 'Protagoras'* and a few others.

PLATO then went to Taranto in southern Italy, where he became a close friend of the Pythagorean statesman and mathematician ARCHYTAS (Ἀρχύτας). From there, PLATO made his first journey to Syracuse, in 388 B.C.E., to the court of the tyrant DIONYSIUS the Elder. But soon, violent differences of opinion emerged and PLATO returned to Athens in 387.

After 367 B.C.E., DIONYSIUS the Younger had followed his father to the throne, and PLATO again received an invitation to the court of Syracuse. PLATO accepted this invitation and arrived in Sicily in the spring of 366. But intrigues at the court robbed PLATO of any influence he might have enjoyed, and he travelled back to Athens in 365. He once again accepted an invitation to the court of Syracuse in 361, but returned to Athens for good in 360.

There, he devoted himself exclusively to teaching at the Academy. He died while writing *("scribens mortuus",* as CICERO passed down) at the advanced age of 80 or 81 in 348 (or 347?). He was buried near the Academy that he had founded - cf. PAUSANIAS: *'Description of Greece'*, Book I, 30.

DIOGENES LAËRTIUS reported in detail about the life of PLATO in the 3rd book of his work *'Life and opinions of famous philosophers'* (*'De vitis, dogmatibus et apophthegmatibus clarorum philosophorum',* written around the year 220). APULEIUS gave a short description of PLATO's life in his book *'Plato and his system of beliefs'* (*'De Platone et eius dogmate'*, written around 160). PLATO's journeys to Sicily were described by PLUTARCH in his biography of DION (see also PLATO*'s* 7th letter).

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2.1 PLATO's views on the teaching of mathematics: *mathesis as anamnesis*

After returning, in 387 B.C.E., from his first trip to Sicily, PLATO (guided by the model of the Pythagorean covenant) founded a school of philosophy on the tree-covered grounds of the sanctuary of the Attic hero HECADEMOS. This sanctuary was located about 2 *km* northwest of the Acropolis. The name of the sanctuary was later transferred to the school, which was thereafter called the *Academy*.[1] Following the Pythagorean dogma that only with the help of mathematics (number theory and geometry) is an insight into the order of the world possible (we wrote about this in Chapter 1), PLATO (according to the legend - cf. D.H. FOWLER, op. cit., p. 200-201) put the following words over the entrance of the Academy:

> Μηδεὶς ἀγεωμέτρητος εἰσίτω.
> [Let no one enter who is unacquainted with geometry.]

The ignorance of the Hellenes in regard to mathematical matters was drastically deplored by PLATO in his book on laws (the *'Nomoi',* Book 7, 819d-e). He wrote that their ignorance robbed them of any appearance as human beings, but rather made them resemble a *"herd of pigs"*.

PLATO never engaged in mathematics in his life, but he always dealt intensively with its epistemological and ontological status, and thereby had a lasting influence on the further development of the field. The dialogue *'Menon',* which he wrote around the years 386/385 B.C.E., was the first work in which he expressed himself on the foundations of mathematics.

In our introductory Chapter 1, we wrote about the *shift of paradigms* in the development of arithmetic and geometry that took place in the first half of the fifth century B.C.E. It was only then that mathematicians realized that neither did the objects they dealt with belong to the world of sense-perceivable objects, nor were they purely linguistic in nature. The fundamental question arose as to what their nature is and how knowledge can be gained about them. PLATO tried to give an answer in his dialogue *'Menon'*.

This dialogue is not initially about mathematics, but rather about the question of whether virtue (ἀρετή, excellence, nobility) can be taught.

PLATO had already dealt with this question some years before under the influence of the Socratic philosophy and had come to the conclusion that it is not possible to say exactly and comprehensively what 'virtue' is, and that, therefore, 'virtue' as such can neither be taught nor learned (cf. his dialogue *'Protagoras'*, 361a). In regard to the concept of learning, at that time, PLATO still understood it to mean *learning by repeated practicing.* (PLATO, in his *'Protagoras'*, used the word διδάσκω = "teaching" in the sense of *teaching on the basis of training*.)

But now, PLATO, using a new concept of learning, succeeded in finding a positive

[1] The Platonic Academy existed for about 300 years in total. In 88 B.C.E., the Roman consul SULLA had it destroyed. A few centuries later, the *New Platonists* rebuilt the Academy. But, in the year 529, it was finally closed by the east Roman emperor JUSTINIAN, because its members wanted to remain faithful to the Hellenistic tradition and not be Christianized. - See also the essay by ALAN CAMERON: *'The Last Days of the Academy at Athens'*, in his book *'Wandering Poets and other Essays on Late Greek Literature and Philosophy',* Oxford 2016, p. 205-245 & 331-335.

answer to the question of whether ‘virtue’ can be taught and learned. This new concept resulted from a reinterpretation of *mathêsis* (μάθησις) as *Anamnêsis,* i.e., *learning by remembering* (ἀνά-μνησις = 'to know again').

The human soul, which, according to PLATO*'s* opinion, is immortal (cf. PLATO*'s 'Politeia',* Book X, 608d), has gotten to know everything that exists before it enters its earthly existence, and only has to remember what it has ‘seen’ in its former existence and what *ideas* (ἰδέαι) it carries within itself. Additionally, ‘virtue’*,* PLATO is now convinced, can be learned by remembering what the soul has already learned before its reincarnation, that is, before its entry into the present human existence, and therefore has *a priori* knowledge of what ‘virtue’ is. That which ‘virtue’ really is cannot (in PLATO*'s* opinion) be described in words by means of an all-encompassing definition, but can be brought into consciousness by reflecting on the archetype (παράδειγμα, model, pattern) of all ‘virtue’, that is, on an archetype that the soul carries within itself (cf. PLATO*'s 'Eutyphron',* 6e-7a).

To prove that all learning is ultimately based on recollection, PLATO uses an example from geometry in his dialogue *'Menon'* (lines 80d-85e). Through clever questioning, SOCRATES prompts a young servant, who had never been taught mathematics, to find the solution to a geometrical problem in his own mind. The point is to prove that a servant who has never been taught geometry can learn geometric facts entirely from within himself, simply by remembering what his soul still knows ‘from before’.

In *'Menon'*, PLATO lets SOCRATES have the conversation with the servant. SOCRATES also considers the soul (ψυχή) to be immortal, and because it has seen everything that is here in the world and everything that is in the underworld, it is not astonishing

> *if it... remembers what it once knew.* (81c-d)

In order to show that this is, in fact, the case, SOCRATES calls in the young servant. The conversation proceeds as follows (82b-85e):

> SOKRATES*: So tell me, boy, do you know that a square is such a figure?*
> *Servant: Yes, I know that*.

SOCRATES then asks the servant how large a square is, the sides of which are doubled. The servant recognizes that it is quadrupled.

> SOCRATES*: So from the twofold side we do not get the twofold square, but the fourfold square?*
> *Servant: You're right.*

SOCRATES then asks how large the side must be if the square is only to be doubled in total. They discuss this at length. Finally, SOCRATES records the result (85a-c):

> SOCRATES*: Does not this line, which goes from one angle to another, cut each of these squares into two equal parts? ... The scholars call it a diagonal; so that if this line here is called the diagonal, then from the diagonal ... the double quadrilateral arises.*
> *Servant: Indeed, Socrates.*

So, at the end of the conversation, the servant has realized that the square of the diagonal doubles the given square. - The servant may now return to his work.

SOCRATES believes that he did not teach the servant, but only questioned him, and that

he (the servant), *'as in a dream'* (85c-d), had brought the knowledge out from within himself. Since it had not been taught to him (in this life), he could only have drawn it from the memory of an earlier knowledge that the soul possessed before its birth.

The theory of recollection (anamnesis) was represented by PLATO in *'Phaidon'* and *'Menon'*. It is based on the old doctrine of the followers of ORPHEUS (οἱ Ὀρφικοί) and the Pythagoreans that the souls of men, before they were born into this world, belonged to another world. PLATO concludes from this that the human soul has known objects and facts in this other world that it forgets at its birth into this world, but which it can remember. The soul has seen the whole of reality. According to PLATO, human learning is only a reminder of the soul of knowledge that it already possesses *'from before'* (*a priori*) and that it only needs to remember.

The Platonic theory of recollection (anamnesis) has been criticized since ancient times. ARISTOTLE argued (*'Prior Analytics',* Book II, 67a 22-27) that SOCRATES directed the conversation and that the young servant learned and acquired knowledge at every step. It only looks as if he was remembering. OSKAR BECKER pointed out (cf. p. X in *'Zur Geschichte der Griechischen Mathematik'* (1965), op. cit.) that SOCRATES drew the diagonal line as an auxiliary line. BECKER remarked:

> *"darin liegt gerade der »springende Punkt«, der eigentlich schöpferische Gedanke der ganzen Betrachtung. Und auf diese Hilfslinie kommt der Sklave nicht »aus der Erinnerung«, sondern der in der Geometrie erfahrene Lehrer Sokrates zeichnet sie ein und teilt dadurch dem Schüler den entscheidenden Gedanken mit."*
>
> [therein lies precisely the "crux", the actually creative thought of the whole argument. And the servant does not come to this auxiliary line "from memory", but Socrates, a teacher experienced in geometry, draws it in and thereby communicates the decisive thought to the boy.]

The objections of ARISTOTLE and BECKER are convincing. In the form put forward by PLATO, the doctrine of recollection cannot be considered well-founded.

But it is remarkable that variants of this doctrine were also repeatedly presented in later times. For example, DESCARTES' doctrine that the basic geometric and arithmetic concepts (straight line, circle, area, space, number, etc.) are present in our soul from birth on, and, in this respect, could be called *ideae innatae*, is one such variant of the Platonic doctrine - we will return to this in Chapter 8.

One can also infer from PLATO's argument, which led to the theory of recollection, that the *a priori* character of mathematical knowledge is asserted in it. We will find this assertion again in different guises, especially in IMMANUEL KANT (cf. Chapter 12).

But first, we want to return to the Platonic theory of recollection and take up again the statement made above, according to which the soul carries an idea within itself of everything it has seen in its former existence. - What is meant by the term *'idea'*?

2.2 The Platonic doctrine of *'ideas'*

Like many pre-Socratic thinkers and poets (EPICHARM, HERACLITUS, PARMENIDES and others), PLATO was convinced that all things, which are perceptible only through the senses, are subject to permanent change. They emerge, blossom and decay again. They never

remain the same. That which changes permanently between becoming and perishing cannot be considered as truly being. That which is uncreated and imperishable, and hence will always remain the same, *can* be considered as *truly being*. Only that which always remains the same is suitable for real knowledge (cf. PLATO in his *'Politeia',* also called *'Republic',* 521d-527c, and ARISTOTLE *'The Metaphysics'*, A6, 987a32-35).

PLATO was convinced that, besides the world of sense-perceptible things (κόσμος αἰσθητός, *mundus sensibilis*), there is also a world of things that can only be perceived mentally (κόσμος νοητός, *mundus intelligibilis*). He suspected that what we experience with our senses in the real world (φύσις) are only fluctuating images of another world of things that truly exist, unchanging and always remaining the same. This led PLATO to put the world of things perceivable through the senses in juxtaposition to a *cosmos of archetypes*. The sense-perceivable things are the *images*[2] of the corresponding *archetypes* (see Plato*'s 'Timáios'* 28a-29d). These archetypes are uncreated, imperishable and always exist in the same way. PLATO called them *ideas* (ἰδέαι). The world of ideas is the realm of that which truly and eternally exists, and, as PARMENIDES had already taught, is only accessible to the mind (νοῦς).

PLATO tried to make his conviction plausible with a parable. This is the famous parable (or allegory) of the cave (we will only reproduce a part of it). In it, PLATO compared human perception with the perception of people who live in an underground, cave-like dwelling.

The allegory of the cave (*'Politeia',* 7th book, 514a-518b). There is a cave, the entrance to which faces the sun. A path leads past it, and people are constantly moving to and fro along it. There are people who live in the cave who cannot see the entrance; but they can see the shadows of people passing by outside on the walls of the cave. They cannot see the people passing by themselves, but they can hear them talking and laughing with each other. Apparently, the cave inhabitants must believe, the sounds are coming from the passing shadows.

PLATO asked the question as to whether our situation in the world is not comparable with the situation of the cave dwellers. The cave dwellers do not know that their perception only refers to shadow-images. PLATO then proceeds to the more fundamental question regarding what our perception refers to. He wants to convince us that we also perceive only shadow-images with our senses, namely, shadow-images of things that belong to a higher sphere of Being. This higher sphere of Being is the world of *ideas.*

The doctrine of 'ideas'. What PLATO called an *idea* (ἰδέα, εἶδος) is the *common image of appearance* of all things that belong to a certain genus or species. This 'common image of appearance' is understood as an independently existing object, and is also called the archetype (παράδειγμα, prototype, model) of the individual appearances (cf. KURT VON FRITZ, op. cit.). The word 'image' is used here only metaphorically.

The word εἶδος is derived from the verb ἰδεῖν (to see, to discern, to recognize the visible form, to know). Originally, it meant only the shape, the outward appearance, the visible form. PLATO gave this word the more abstract meaning of *"common image of appearance*

[2] PLATO uses the word εἰκών (=image). In the exact sense of the word, εἰκών is that which is the same as that which is perceived.

of all objects of a genus or a species". According to PLATO, an *idea* is something that cannot be seen with the eyes, but that the soul of a human being can look at (cf. PLATO: *'Euthyphron'*, 6e-7a). Again, the verb "to look at" is meant here in a metaphorical sense.

From this description, it is already clear that the Platonic term *'idea'* is not identical to what is usually called an *'idea'* in English. In particular, PLATO understands *'ideas'* as things (objects) whose existence is imperishable.

In the world of *'ideas'*, there is the *'idea'* of a tree, the *'idea'* of the Good, the *'idea'* of the Beautiful, and just as much the *'ideas'* of all basic mathematical objects, namely, the *'idea'* of a unit, the *'idea'* of a geometric point, of a straight line, of a plane superficies, of a circle, of a square, of a cube, etc.

2.3 The world of mathematical objects

According to PLATO, the realm of mathematical things lies between the world of things perceivable through the senses and the world of *ideas* (cf. ARISTOTLE *'Metaphysics',* book A, chapt. V, 987b15-20). The intermediate position is justified as follows: The objects of mathematics are not the ideas themselves, because there is, for example, only one idea of a circle, but many individual circles (all these mathematical circles are images of the idea of a circle), and there is only one idea of a unit, but many individual units, etc. Mathematical objects, however, are, as in the case of ideas, only mentally perceivable (and thus different from things perceivable through the senses).

For the recognition of facts that are true in the world of mathematical objects, one is hence in need of *anamnesis* (ἀνάμνησις, recollection), *dianoia* (διάνοια, logical thinking) and *noesis* (νόησις, mental perception).

Just as PLATO distinguishes between the mathematical circles that are only mentally perceivable and the circles on paper or in the sand, which are perceivable through the senses, he also distinguishes (in his dialogues *'Philebos',* 56d-e, and *'Politeia',* 526a-b) between the mathematical numbers that are only there for the intellect and the numbers that *"the merchants and the many other people"* deal with in everyday life.

In daily life, *"unequal units are always counted together"*, e.g., twenty people (who are all different from each other), or ten apples (again, which are all slightly different from each other), etc. In scientific arithmetic, i.e., pure mathematics, on the other hand, one operates with pure numbers, and here the numbers are sets (or multitudes) of indistinguishable units. These units cannot be found in the real natural world, but can only be thought of. These numbers of pure arithmetic belong to the world of mathematical objects, just like the expansionless points, the breadthless straight lines and circles, etc., of pure geometry.

It is noteworthy that PLATO did not identify the numbers with the numerals, as was always the case in older cultures. PLATO understood numbers as objects (things) different from their names and different from their notations (cf. PLATO*'s* Seventh Letter, 342b-c). In his opinion, numbers are things that *do not belong to the world of things perceivable through the senses,* but to the *mentally perceivable world of ideas and their images in the realm of Mathematics*.

Likewise, the objects of scientific geometry are not sense-perceivable. About the role

of geometric drawings, PLATO writes, in the *'Republic'* (i.e., *'Politeia'*, VI, 510d-e) – correspondingly translated:

> *"... and also that they make use of the visible figures and always refer to them in their discourses, regardless of the fact that they do not deal with them, but with their ideal counterpart to whom they resemble; and perform proofs not for the sake of what they draw, but for the ideal square and the ideal diameter, and also everywhere else. That which they themselves reproduce and draw, of which there are also shadows and images in the water, they make use of as images, but they always seek to recognize that, which cannot be seen otherwise than with the eye of the mind."*

The drawings only accompany the proofs; they can suggest facts that are often difficult and cumbersome to express in words. In Platonic geometry, the function of a diagram is only to represent in drawings what should actually be communicated in words (and concepts) that refer to ideas.

If lines can be constructed with the help of mechanical aids, then this may be useful for (practical) *geodesy*, but in the Platonically conceived (theoretical) *geometry*, such constructions are not permitted.

For PLATO, curves are permissible if they have ideal descriptions, i.e., if they can be described exclusively by relying on means that are available in the world of geometric ideas. Also, the insertion of straight lines of a given length and a given direction into a geometric diagram (νεῦσις, *inclination, gerichtete Einschiebung*) is not permitted in a geometry as conceived by PLATO, because the final position of the inserted line cannot be described in advance by relying only on means that are available in the world of ideas. This requirement, for instance, is not met by many mechanical constructions (e.g., the Archimedean trisection of an angle, the Archimedean construction of a regular heptagon, the construction of two mean proportionals for doubling a cube as done by HIPPOCRATES of Chios, etc.). Such constructions are very useful in practice, but do not belong to theoretical geometry as PLATO conceived it. For this, the following quotation from PLUTARCHUS may serve as a good record:

> *In particular it is geometry, which ... redirects the mind freed from sensuality and gradually purified. This is why Plato also rebuked Eudoxos, Archytas and Menaichmus for trying to bring back the doubling of the cube to mechanical instruments and devices .... It is precisely because of this, he said, that the usefulness and merit of geometry is completely lost when it returns to sense-perceivable things instead of soaring up and dealing only with the eternal incorporeal images....*
> PLUTARCHUS*: 'Table Talks' (*Συμποσιακά, VIII, 2); and PLUTARCHUS*: 'De vita Marcelli', 14.9 -12.*

The Platonically-conceived geometry is about the incorporeal points, lines, surfaces and bodies. PLATO himself writes, in his *'Republic'* (i.e., *'Politeia',* 527b7):

> τοῦ γὰρ ἀεὶ ἡ ὄντος γεωμετρικὴ γνῶσίς ἐστιν.
> [... because geometry is the knowledge of the eternally Being.]

In conclusion, we should like to state that, according to PLATO, the world of things is divided into four object areas. It can be represented schematically as follows (cf. PLATO*'s 'Politeia'*, 509c-511e, and ARISTOTLE*'s 'Metaphysics'*, 1st book, § VI):

Mundus intelligibilis, νοητὸν γένος (the field of eternal being: οὐσία)	(1) the world of ideas (2) the world of mathematical objects (images of the ideas)
Mundus sensibilis, ὁρατὸν γένος (the field of becoming: γένεσις)	(3) the world of objects perceivable by sense (4) the world of shadow images of the objects perceivable through the senses

2.4 The construction of a mathematical theory according to PLATO

In the preceding sections, we have discussed the *ontological status* of mathematical objects in some detail. We have found that, according to PLATO, the basic mathematical objects are *images of ideas*, and the sense-perceivable geometric diagrams are only *imitations* of the corresponding mathematical objects. Also, the numbers of sense-perceivable objects are only *imitations* of the corresponding mathematical numbers, which are (finite) collections of units. All of these mathematical objects are only mentally perceivable. [Notice that this will also explain, in a very simple way, why mathematics is applicable in physics].

We will now turn to the question of how to obtain knowledge of the facts that are true in the realm of mathematical objects. How does one gain insights into that which is valid there? What is the *epistemic status* of arithmetic, and also that of geometry?

Mathematical statements are always assertions. They assert that certain mathematical objects stand in certain relationships to each other. PLATO assumes that the realm of basic mathematical objects is a well-established realm and that everything is fixed there, i.e., nothing depends on any preconditions. We get insights into the world of mathematical objects through our mind (νοῦς), through our thinking (διάνοια), and also through our soul (ψυχή) by remembering earlier knowledge.[3] From these remarks, we can draw the following conclusions, which, however, cannot be found in this explicit form in PLATO's surviving works.

In the process of constructing a *mathematical theory*, such as geometry, one must first mention the basic objects and describe them in such a way that the *ideas* of which these objects are representations (or imitations) become sufficiently clear and can be grasped by the human mind (the *nous*). For example, the basic concept of a *straight line* has to be described to the extent that the intuitive content of this concept is sufficiently clear to the mind. According to PLATO, it is not necessary to place *axioms, postulates* or *hypotheses* in the outset of a mathematical theory and demand their acceptance, because the truth of such axioms, postulates or hypotheses has to be evident, and that which is evident must not be mentioned. The human soul should be able to remember its prenatal knowledge, and thus

[3] In the appendix to the *'Nomoi'* (the so-called *'Epinomis'*), it is written that the gods had given us the sciences (4, § 11), that geometry was a divine invention (12, § 45) and that the gods would also guide us in the knowledge of arithmetic truths (4, §11). Such dogmata are found nowhere else in the writings of PLATO, But they are dogmata that the Stoics advocated. (We will report on this in Chapter 8.) For these and certain other reasons, it has been assumed since antiquity that the *'Epinomis'* cannot be counted among the genuine writings of PLATO.

be able to confirm the facts expressed in the axioms, postulates or hypotheses. Thus, as soon as all basic *ideas* are clearly stated in the outset of a theory, one can start exploring the realm of objects of the theory in question and also constructing further objects out of the basic ones. In this way, one can start proving theorems without any further ado (cf. PLATO*'s 'Politeia'*, Book VI, 510 c-d).

2.5 Of *ideas*, *notions* and *concepts*

In the preceding paragraphs, we have described in some detail the views of PLATO with respect to the ontological, as well as the epistemological, status of mathematical objects and theories. In some respects, they are impressive, and many people have been impressed and fascinated by them. The fact that both arithmetic objects and geometric objects are, in some way, 'removed' from the 'real world' is beautifully expressed here, as is the *a priori* character of mathematical knowledge.

PLATO believes that mathematical objects are not created and don't perish; their existence is eternal. They are, hence, not only *onta* (ὄντα), but *ousiai* (οὐσίαι). Here, τὸ ὄν means, in general, 'Being'. The Platonic οὐσία, however, is *'continuous, eternal Being',* a manner of Being that is removed from space and time (cf. KURT V. FRITZ, op. cit., p. 73).[4] For PLATO*,* mathematical objects are immaterial objects that do not belong to our spatio-temporally structured world.

But how do we get insights into this world of mathematical objects? This is an eminent epistemological question. The answer, given by PLATO, is this: it is basically the mind (νοῦς) of a human being that is able to 'look' at the *ideas*, and thereby understand what is true for them.

The rudiments of epistemology can already be found in the works of PLATO. In his dialogue *'Theaitetus'*, he discusses the question as to whether *knowledge* (ἐπιστήμη) is the same as *perception through the senses* (αἴσθησις), or perhaps the same as *notorious opinion* (ἀληθὴς δόξα), etc. The process of sense-perception is nicely explained as follows (191c7-191d2). Imagine that there exists in the mind of a human something like a *'block of wax'* (κήρινον ἐκμαγεῖον), and that everything that is perceived by the senses will be imprinted by the soul on this block of wax. According to PLATO, this aptitude of the soul is a gift from Mnemosyne, the mother of the muses, and allows the soul to reflect about that which the senses have registered. However, in order to achieve knowledge, a contribution from the mind (νοῦς) is also necessary, namely, the ability to recognize all the 'ideas' that correspond to the impressions brought about by the senses. (Compare this with KANT's notion of *'pure intuition',* cf. Chapter 12, section 12.7.)

PLATO does not expand his work with the formation of *'concepts',* but instead introduces objects that he calls 'ideas'.

The terms ἰδέα and εἶδος are used by PLATO to signify the *common image of appearance* of all things that belong to a given genus or species. This *common image of appearance* is

[4] The Term ουσία is derived from the verb εἶναι (to be). The suffix ία indicates a class-formation. In fact, ουσία is the essence of an object, i.e. the class of all those properties of the object that eternally belong to it.

an archetype, or prototype, which PLATO understood as an object that exists independently of its images or imitations. But notice that the word *'image'* is used here only in a metaphorical sense. Hence, an *idea* is just an object that is determined by that which is common to all things that belong to the genus or species under consideration. The way in which it is determined, however, was not made clear by PLATO. But it is clear that, for the *Platonic doctrine of ideas*, it is the mind (νοῦς) that receives insights into the realm of ideas. (Again, the word 'insight' is used here only in a metaphorical sense.) The mind is able to use these insights in its thinking.

In the century after PLATO's death, ARISTOTLE and the Stoic philosophers (ZENO of Citium, CHRYSIPPUS and others) transformed the Platonic *doctrine of recollection* (anamnesis) and the *doctrine of ideas* into an epistemological theory.

An important step was taken by ARISTOTLE in his work *'Peri Hermeneias'* (Περὶ ἑρμενείας), also called *'De Interpretatione'*. In this treatise, the main subject is the relation between sensory perception and mental perception, the relation between language and thought.

A further step was taken by the *Stoic philosophers*. Their theory has been handed down to us in a number of fragments, in particular, in a well-written fragment credited to AËTIOS (cf. fragment 83 in J. V. ARNIM, or fragment 277 in K. HÜLSER, op. cit.). Note that this fragment can also be found in book 4, section 11, of a treatise that, for a long time, was attributed to PLUTARCHUS (op. cit., vol. 21, pp. 198-199), called *'Placita philosophorum'*. Hence, it was well-known to most philosophers post-antiquity. Now, it is almost unknown. An account of this fragment follows.

It is argued in that fragment that, when a human being is born, the memory of its soul (more precisely: its ἡγεμονικὸν) can be compared with an empty piece of paper (a *tabula rasa*) upon which the soul registers, during its life, all of its *ennoiai* (ἔννοιαι), i.e., everything of which it becomes aware and everything that constitutes that which it *knows*. There are two different kinds of registration.

The first kind is caused by sense-perception. When an object is perceived through the senses, there remains a reminder (μνήμη) of it in the soul. Out of the reminders of many objects of the same genus, the soul will produce an impression that contains all of those aspects that all of the reminders have in common. The Stoics called this *prolepsis* (πρόληψις), or, in Latin: *praenotio* (pre-notion).

The second kind of registration rests on mental reflections on that which the soul already bears within itself. These registrations, which are created consciously, are called *ennoëmata* (ἐννοήματα). Their contents can be expressed in language, but they do not necessarily represent genuine objects of the real world. Each of these registrations is, hence, just a *phantasma dianoias* (φάντασμα διανοίας), i.e., an imagination, or a 'Gedankenbild'.

Both kinds of registration are called *ennoiae* (ἔννοιαι), i.e., knowledge.

This fragment, credited to AËTIOS, indicates that a fundamental change with respect to epistemology took place in the century following PLATO's death. The role that was played by the Platonic *ideas* was subsequently undertaken by the *ennoiai*. Their place of existence is not an independent world of objects (such as the Platonic realm of ideas), but the human mind (or soul) itself.

According to CICERO (*'Topica',* 7,31, see also his *'Academica',* editio prior, Book II, X,§30), *'notio'* is a good translation of the Greek word *ennoia* (ἔννοια) into Latin. Therefore, *'notion'* (or *'knowledge'*) seems to be a good translation of ἔννοια into English. The word *'notion',* however, should be understood in the sense of *'thought'* (*'pensée'* in French, *'Innewerdung'* in German), and not in the sense of *'concept'* (i.e., *'compréhension de l'idée'*, as explained in the *'Logique de Port-Royal'* (Paris 1662) by ANTOINE ARNAULD and PIERRE NICOLE).

2.6 Concluding remarks

It is remarkable that PLATO initiated such a development with his doctrine of *ideas*, ultimately leading (with the work of A.M.T.S. BOËTHIUS, RENÉ DESCARTES, JOHN LOCKE, ANTOINE ARNAULD, PIERRE NICOLE and others) to what we now call a *'concept'* (*'conceptus'* in Latin, *'Begriff'* in German). We will return to this subject in Chapter 9 and in the Epilogue.

In conclusion, although we have to note that PLATO did not succeed in clarifying the ontological and epistemological problems that mathematics raises, we can emphasize that he was successful in naming and tackling the most important questions and problems that affect the philosophy of mathematics. Before him, no one had recognized and discussed these problems. PLATO*'s* thoughts have stimulated mathematicians and philosophers up to the present day to re-examine the great questions and problems of the ontology and epistemology of mathematics. In much of their work, the Platonic answers can be found in a modified form.

We have already mentioned some of these answers (given by DESCARTES and KANT) above. The view that mathematical objects have a being that is independent of our human thinking has had a strong influence in the so-called *Platonism* of the 20th century (in the sense of PAUL BERNAYS) - see Chapter 16.

We will discuss these and other after-effects in modern times in detail later on. PLATO initiated the practice of thinking about the foundations of mathematics and has influenced it considerably up to the present day.

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Chapter 3
The Aristotelian Conception of Mathematics

'l maestro di color che sanno
[ARISTOTLE, the master of those who know]
DANTE ALIGHIERI*: La Divina Commedia, I, IV,131.*

ARISTOTLE (Ἀριστοτέλης) was born 384 B.C.E., in Stageira (in the border area between Thrace and Macedonia). He entered PLATO*'s* Academy in 367 and remained a member until PLATO*'s* death in 348/347. In 343, he became a teacher at the Macedonian royal court of the then-13-year-old ALEXANDER (later called 'the Great'). In 336, he returned to Athens and, soon after, became head of the Lycëum (Λύκειον), a gymnasium sacred to Apollo Lyceus. ARISTOTLE delivered his lectures on philosophy while walking up and down (περιπατῶν) along the shady walks that surrounded the Lycëum. The members of this school were therefore called 'Peripatetics'.

After ALEXANDER*'s* death in 323, ARISTOTLE was accused of godlessness in Athens. He fled to Chalkis on Euboea, where he died shortly afterwards (approx. in the year 322 B.C.E.).

In the Aristotelian philosophy of mathematics, the ranking of things with respect to their *Being* is reversed in comparison with PLATO*'s* philosophy. While PLATO*'s* 'ideas' have the highest *Being* (ὄντως ὄν), and things, which are only perceivable through the senses, have a derived *Being* of lower order, ARISTOTLE sees every single thing of the world that can be experienced with our senses as a carrier of *Being*, whereby mathematical objects only have a derived *Being*.

Cognition is not obtained by looking into the Platonic world of ideas, but by sense-perception – see our remarks on that subject in Chapter 2, section 2.5. As a result of this conviction, ARISTOTLE completely rejected PLATO*'s* doctrine of ideas.

ARISTOTLE spoke about the foundations of mathematics mainly in the following works:

'Posterior Analytics' (Ἀναλυτικὰ ὕστερα, *Analytica posteriora*),
'The Physics' (Φυσικὴ ἀκρόασις, *Physica*),
'The Metaphysics' (Τὰ μετὰ τὰ φυσικά, *Metaphysica*).

But he also addressed them in the *'Topics'* (Τοπικά, *Topica*) and the books *'On the Heavens'*

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(Περὶ οὐρανοῦ, *De Caelo*) and *'On the Soul'* (Περὶ ψυχῆς, *De anima*), which have occasional references to matters of mathematics.

In these writings as a whole, ARISTOTLE gives a very detailed account of the fundamental problems of constructing mathematical theories. We begin with a presentation of his views on the general concept of a theory and then turn to the concept of a mathematical theory.

3.1 The Aristotelian concept of a scientific theory

ARISTOTLE writes about the construction of a scientific theory in the two books of his *'Analytica posteriora'*.

According to ARISTOTLE, all sciences (scientific theories) study the sense-perceivable world. They may not study the whole world, but they do at least study the objects of a certain *species* or *genus*. They may not study the objects of such a species or genus in every respect, but they do study them in at least certain respects.

According to ARISTOTLE, *physics* studies the objects of the world only from the point of view of movement, *music* studies objects only from the point of view of sound and *medicine* studies living Beings only with respect to health. *Geometry* studies the objects of the world only from the point of view of the shape or form of their extension, i.e., their straightness, sphericalness, circular form, cube form, etc.

Mathematicians therefore do not differ from physicists in regard to the realm of objects that they study, but only in the way that they examine them.

ARISTOTLE claims that, in every science, first of all, the class of objects (the genus) has to be indicated, i.e., the totality of objects that a particular science should deal with (cf. also the 6th book of the *'Metaphysics'*, 1025b8-9) and in what respect the objects should be investigated (i.e., *'as what'* or *'in what respect'* (ᾗ, qua) they should be investigated).

After that, all *principles* [1] (ἀρχαί, principia) must be stated. The principles are divided into (1.) general (κοιναί) and (2.) specific (ἴδιαι):

(1.) ***The general principles*** are the *generally valid statements*, as well as the *generally accepted opinions* (κοιναὶ δόξαι, *'Metaphysics'* III 2, 997a20-22), sometimes also called *axioms* (ἀξιώματα). These are statements that are valid in all theories. *Aristotle* gives, as examples, some equality axioms , for example *"if a = b and c = d , then a – c = b – d"* (*'Posterior Analytics', 76a41*, and *'Metaphysics'* XI 4, 1061b18-24), and also the Aristotelian syllogisms of predicate logic. But ARISTOTLE also mentions the principle that every statement formulated in the language belonging to the theory should be either true or false (*'Posterior Analytics'* 71a14, and *'Metaphysics'* III 2, 996b29-30) and the principle that a statement should not be both true and false at the same time (*'Posterior Analytics',* 77a11).

(2.) ***The special principles*** are the *theses* (θέσεις). These are the principles peculiar to the individual sciences. They fall into two classes, depending on whether they *explain* something or *claim* something.
(2a) The *theses* in which something is explained are called *definitions* (ὅροι). They

[1] *principium = id quod primum cepit* (what is grasped first).

form the actual basis of the discipline to be investigated.
(2b) The *theses* in which it is claimed that something is a fact are called *hypotheses* (ὑποθέσεις).

(2a) The ***definitions*** shall assert something about the objects under investigation, namely,

(∗) *what they are,*
(∗∗) or *why they exist,*
(∗∗∗) or *why other things exist, from whose existence one may gain information about the objects to be examined.*

ARISTOTLE distinguishes three different types of definition[2] (cf. *'Analytica posteriora',* 2nd book, Chapter 10, and *'Topica'*, 6th and 7th book).

To the first type belong the so-called '*definitiones essentiales*'. These are statements in which the *essence* that is common to all things belonging to a certain species of things is defined. According to ARISTOTLE, a definition of the essence (ὅρος οὐσιώδης) consists in the declaration of the next higher *genus* (the *genus proximum*) and the species-forming difference (the *differentia specifica*) that separates it from other species belonging to that genus. Here, 'essence' is the translation of the Latin word *essentia,* which, again, is the translation of the Greek word οὐσία. The definitions of the first type, hence, contain information related to the question: 'what is this thing', *quid sit res.*

The second type is the *causal definition* (αἰτιώδης ὁρισμός). Such a definition contains information about the causes for the existence of the defined object: 'why it is', *cur sit res*.

The third type of definition contains information about the causes of the existence of other things, from which the essential properties of the thing to be defined can be inferred in a syllogism.

(2b) ***The hypotheses*** are either directly insightful statements or statements that can be obtained by *induction* from directly insightful statements (cf. *'Posterior Analytics'* I 18, 81a38-81b9 & II 19, 99b15-100b16). In any case, they must be true propositions (*'Posterior Analytics'* I 2, 71b25-26).

So, the word *'hypothesis'* (ὑπόθεσις) is used here in a rather different way than it is used today. The Greek word ὑπόθεσις is formed from ὑπό and τίθεσθαι (cf. Á. SZABÓ, op. cit., p. 310). It actually means something like 'basis' or 'underlying facts'. The word is also used in this sense in ancient tragedies and plays, in which the 'hypothesis' is placed in front of the text and indicates the underlying matter in short form (often in a way that differs from the tradition).

ARISTOTLE was the first to speak of *induction* (ἐπαγωγή). With it, he denoted the 'ascent from the individual to the general' (*'Topica'* I 12, 105a13-14 & VIII 1, 156a5-6), but did not say exactly what he meant by this (cf. the comment by WOLFGANG DETEL in his edition of *'Posterior Analytics',* op. cit., Vol. 2, pp. 844-846). In order to avoid misunderstandings, it must first be said that the Aristotelian concept of induction is different from today's

[2] The so-called *nominal definitions*, in which the meaning of a concept is reduced to the meanings of other concepts, are not relevant here. Such definitions can be eliminated and, hence, are basically dispensable.

common concept of (incomplete) induction, which goes back to FRANCIS BACON (1561-1626).

ARISTOTLE meant, with the word '*induction*', the transition from a statement $\Phi(c)$ to the universally valid statement $\forall x\Phi(x)$. Here (in today's terminology), x is a variable that runs through a class $\mathcal{K}$ of objects (a *genus* or a *species*). He only allowed this transition if the proof that the object c has the property Φ is also transferable to any other object d of the class $\mathcal{K}$, i.e., if the general can be grasped through the particular (cf. *'Posterior Analytics'* I 2, 71a8-9 and II,19, 100a3-100b5). CHRISTOPH SIGWART (op. cit., volume 2, § 93, 17, pp. 436-438) called this type of induction *"generalisierende Induktion"* (generalizing induction) [3] - see also KURT VON FRITZ, op. cit., 1964 and ULRICH FELGNER, op. cit., 2012.

3.2 The Aristotelian Apodeixis

From the principles and the definitions, which are stated at the outset of the exposition of a scientific theory, one obtains, through '*proof*' (ἀπόδειξις, lat.: *demonstratio*), further valid statements. In the proofs, no further presuppositions may be used. On the basis of the presupposed principles (the universally valid statements, the definitions and the hypotheses), the truth of the conjectured assertion will be established in a sequence of logically correct conclusions, which, however, must all have the form of syllogisms.

From a *scientific theory*, ARISTOTLE demands that the general principles and the specific hypotheses all be true (ἀληθεῖς). Proofs should therefore lead from true statements to true assertions.

In order for a *scientific theory* to differ from a collection of mere empirically observed statements or mere (unfounded) statements of facts, ARISTOTLE requires that, in a proof, conclusions be drawn not only from statements, but from 'good reasons' for the truths of the respective statements. These justifications must, of course, appear among the theses of the theory.

ARISTOTLE thus distinguishes between simple proofs, in which conclusions are drawn only from statements '*that*' (*hoti*, ὅτι, *quod*) a particular object behaves in a certain way, and scientific proofs, in which conclusions are always drawn from the causes as to '*why*' (*dihoti,* διότι, *quare, cur*) it behaves in a certain way (ARISTOTLE, *'Posterior Analytics'*, I 12, 78a23). A scientific proof must therefore be based on the reasons (causes) that show *why* the respective alleged facts are valid. The possible reasons (or causes) are structured schematically by ARISTOTLE as follows.

The four-cause scheme. The word *'aitía'* (αἰτία) used by ARISTOTLE can be translated as 'reason, cause'. In Latin, the translation '*causa*' is common. The term 'cause' is explained by ARISTOTLE in the 1st book of the *'Metaphysics'*, Chapters 2 & 3, and in the 1st book of the *'Physics',* Chapter 2 (see also the 2nd book, Chapter 10, of the *'Posterior Analytics'*). He distinguishes four types of cause:

- the *'causa materialis*' (the *material cause*): here, the cause is that which causes the creation of the object and the material of which it consists;

[3] The 'Theorem on Constants' of modern mathematical logic is related to this Aristotelian 'generalizing induction' - see, for example, JOSEPH SHOENFIELD*: 'Mathematical Logic'*, 1967, p. 33.

- the *'causa formalis'* (the *cause of form*): here, the 'essence' (εἶδος καὶ τὸ παράδειγμα) of the object is given as the cause;
- the *'causa efficiens'* (also called *causa agens*, *cause of action*): here, the cause is that which causes a change;
- the *'causa finalis'* (the *purpose cause*): here, the cause is stated as why or wherefore the object exists.

In order that mathematics be a 'scientific theory' in the sense of ARISTOTLE, the proofs it gives must be proofs in which conclusions are drawn not from the '*that*', but from the '*why*'. Thus, it is not sufficient just to draw conclusions from certain facts that are either claimed or demanded/presupposed; we must draw our conclusions from the causes of those facts.

Of the four possible causes (*causa materialis, causa formalis, causa efficiens, causa finalis*), only the *causa formalis* is apparently relevant in mathematics. The word εἶδος is usually translated into Latin as *figura, forma, species,* and therefore in English as '*figure, form*'. This word εἶδος does not refer to the '*form*' perceivable through the senses, but to the spiritually perceivable form, i.e., the *essence* of the object, i.e., its definition of the essence (ὅρος οὐσιώδης). This is the reason why, in scholasticism, the cause mentioned was called *causa formalis.* The *causa formalis,* hence, consists of the naming of all species to which the object under consideration belongs, i.e., the definition of its essence.

That, in mathematics, only the *causa formalis* is relevant is emphasized by ARISTOTLE himself in his *'Physics'*, Book II, Chap. 6, 198a16-18. It says there:

> οἷον ἐν τοῖς μαθήμασιν· εἰς ὁρισμὸν γὰρ τοῦ εὐθέος ἢ συμμέτρου ἢ ἄλλου τινὸς ἀνάγεται ἔσχατον.
>
> [As in mathematics, where the conclusions ultimately depend upon the definitions (of the concepts) of straight line, or commensurability, or similar things.]

The 'definitions' referred to here are the 'definitions of *essence*' (*definitiones essentiales*). Thus, the Aristotelian requirement here is that, in a scientifically operating mathematical discipline, one may only rely in one's proofs on the *causa formalis*, i.e., only on definitions, namely, on nominal definitions and definitions of the essence of the objects under consideration.

ARISTOTLE is convinced that arithmetic and geometry, if constructed according to his proposals, form scientific theories. Whether Euclidean geometry and Euclidean arithmetic are 'scientific theories' in the Aristotelian sense was intensively and controversially discussed in the Renaissance. We will report on this in Chapter 7.

3.3 The ontological status of mathematical objects

ARISTOTLE spoke about the nature of numbers and geometric objects in his lectures on *'Physics'*, in his books *'On the Soul'*, *'On Heaven'* and in the collection of independent texts that have been handed down to us under the later added title *'Metaphysics'*.

In the 13th book of the *'Metaphysics'*, ARISTOTLE discussed the different views on the ontological status of the basic objects of arithmetic and geometry that were being investigated in the philosophical schools of the time. The question was whether the

mathematical objects exist *on* (or *in*) the sense-perceivable objects of the world or independently of them. Do mathematical objects owe their *existence* to the sense-perceptible objects of the world or do they, in fact, have an independent existence, as the Platonic philosophers taught?

ARISTOTLE wanted to lay a foundation for arithmetic and geometry in which (unlike in PLATO) a reification (hypostasis) of the basic concepts is avoided. If, as PLATO did, one reifies the basic concepts, then the problem arises as to how it comes about that the basic concepts participate in their reification. ARISTOTLE hoped to avoid this problem by considering as mathematical objects those sense-perceivable objects to which certain mathematical attributes belong. For example, if we recognize, with our *senses*, an object that has 'length', this object can be considered, in our *thinking*, *'as'* (ᾗ, *qua*) a geometric line without breadth, because we are able to ignore all its other properties, just keeping in mind its length.

This was his approach to the foundations of geometry and, similarly, to the foundation of arithmetic. - What this means will be explained in the following. We will also see whether he was successful in constructing geometry and arithmetic.

ARISTOTLE does not want to question the fact that mathematical objects are, in some sense, things that *exist*. He does not want to represent a kind of pure *nominalism.* He says that geometry and arithmetic can only be sciences if they deal with existing objects. Thus, he assumes that arithmetic and geometry refer to things that *exist* (they are ὄντα) and that the only relevant question regards in which way they 'are', i.e., which ontological status they have. He writes (*'Metaphysics',* XIII, 1, 1076a36)

> ὥσθ' ἡ ἀμφισβήτησις ἡμῖν ἔσται οὐ περὶ τοῦ εἶναι ἀλλὰ περὶ τοῦ τρόπου.
> [Our disagreement, then, will not concern the existence (of mathematical objects) but the mode of existence.]

ARISTOTLE*,* after a long back and forth, came to the conviction that mathematical objects cannot exist in complete separation from sense-perceivable objects, but also that they are not sense-perceivable objects themselves. However, in some weak sense, they must be separated (or detached). In order to be able to describe this kind of 'separated Being' more precisely, ARISTOTLE uses the two words '*aphairesis*' and '*chorismós*'*:*

> ἀφαιρεῖν (take away, subtract, remove, erase, subtract the unessential, abstract) - and
> χωρίζειν (separate, sever, loosen a bond, segregate).

3.4. Aphairesis (Ἀφαίρεσις)

Aphairesis (the taking away) describes (in general) the process of not taking account of some of the properties of a concrete object, for example, that it is made of wood or has a certain colour. For example, in the *'Analytica posteriora'* (74a33-74b1), when a triangle made of bronze is mentioned, the process of not taking account of its bronze nature is called 'aphairesis'. (The word is derived from the verb αἱρέω (= to take away).)

'Aphairesis' is usually translated as 'abstraction'.[4] But, if one translates 'aphairesis' in

[4] It was A.M.T.S. BOËTHIUS who proposed this translation into Latin. Notice that both terms, ἀφαίρεσις and 'abstractio', are Verbalabstrakta.

this way, there can be confusion. In order to avoid this, it should be emphasized that ARISTOTLE does not mean that new (abstract) objects are created in the process of abstraction, but only that - when looking at concrete objects - some of their properties should be left out of consideration. [5] JULIA ANNAS (op. cit., p. 135) has very aptly described this kind of 'abstraction' as *"gezielte Unaufmerksamkeit"* (deliberate inattention).

ARISTOTLE himself does not give an exact description of the process of aphairesis. In his *'Metaphysics'* (Book XI,3, 1061a29-b2), he only gives the following hints:

> *"And just as the mathematician makes a study of abstractions (for in his investigations he first abstracts everything that is sensible, such as weight and lightness, hardness and its contrary, and also heat and cold and all other sensible contrarieties, leaving only quantity and continuity – sometimes in one, sometimes in two and sometimes in three dimensions – and their affections (τὰ πάθη) qua quantitative and continuous, and does not study them with respect to any other thing; and in some cases investigates the relative positions of things and the properties of these, and in others their commensurability or incommensurability, and in others their ratios; yet nevertheless we hold that there is one and the same science of all these things; viz. geometry), so it is the same with regard to Being."* (Translation by HUGH TREDENNICK, The Loeb Classical Library, London 1977.)

Since ARISTOTLE himself does not give an overly precise description of the process of *aphairesis*, we will try to be a little more precise about what should be left out of consideration in this process.[6] We think that the process of *aphairesis* should lead to a sense-perceptible object belonging to a certain genus of things being treated as a *generic object* of that genus, whereby all properties that do not belong to the definition of the genus are left out of consideration, i.e., are literally 'taken away'.[7]

So, the process of *aphairesis* is only explained for things *relative to the genus* they belong to!

If, for example, the genus (or species) 𝔊 of all circular objects of the world of sense-perceivable objects is considered, then the usual concept of the circle (as an object that seems to be drawn in a full revolution with a compass) is *the general* (τὸ καθόλου) that

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[5] The word 'abstraction' is also used to describe the process of taking something away (or leaving something out of consideration) from a perceived object in the course of thinking about it, *and* to lift out what remains as something new that exists independently. Such a lifting out of something that exists independently does not take place in ARISTOTLE. But it does take place in the modern age from the 17th century onwards (J. LOCKE) and it has become the common view since then (cf. Chapter 9).

[6] The paraphrase of the term *aphairesis* given by ARISTOTLE is not precise enough. If, for example, pencil strokes or ink strokes are to be considered 'as' lines in the sense of geometry, then it is not enough to say that "*all sense-perceptible properties are omitted and only the quantitative and the ... continuous* should *be left"*, because the pencil lines always have a certain length *and* width, and one has to be more precise about what should be left unnoticed. One must obviously refer to the respective genera to which the things belong.

[7] An object is called *generic* if it is a general, nameless, typical object of a *genus* and does not have any additional properties that bring it into prominence. In this sense, we speak of *generic points* in algebraic geometry, *generic sets* in set theory and *generic types* in model theory. In the commercial field, *generic products* are referred to as such if they only contain a reference to the genus of the product, but not the name of the manufacturing company.

characterizes the genus 𝔊. According to ARISTOTLE, this *general* concept of the circle is realized in all objects belonging to 𝔊. In the process of aphairesis (relative to 𝔊), all properties of objects that belong to 𝔊 are faded out, with the exception of the defining property of 𝔊. An object *G*, which belongs to 𝔊, is then considered *'as'* (ᾗ, *qua*) a circle in the mathematical sense, even if *G* were to have a certain thickness and irregularity in the sense-perception. Thus, all properties of *G* that are responsible for the deviation from the ideal form are also faded out.

In his book *'De Caelo'* (268 a 1-6), ARISTOTLE says that, in mathematics, one can also regard the boundary points, borderlines, surfaces, etc., of sense-perceivable bodies *'as'* points, lines, surfaces (in the mathematical sense), provided that *everything* sensorial, such as, e.g., gravity, hardness, temperature, etc., is disregarded, or, more precisely - as explained above - if these objects are regarded as generic objects of their respective genera (i.e., the genus of points, the genus of lines, etc.).

A fine example has been handed down from antiquity. SEXTUS EMPIRICUS reported (in his books *'Against the Geometers',* § 57, and *'Against the Physicists',* Book I, § 412) that, according to the Aristotelian view, a straight-line wall may also be regarded *as* a straight line, and, in the sense of geometry explicitly, *as* a line without breadth, because it has length and belongs to the genus of straight-line things. We add, however, that the same wall can also be regarded *as* a surface and also *as* a cuboid, etc., since it also belongs to all the corresponding genera.

In a geometric investigation, it is therefore always necessary to indicate *what* the individual components of the configuration under investigation are to be regarded *as,* since this is usually, but not always, clear from the outset.

The objects that mathematicians look at are, at first glance, bodies that can be perceived by the senses. In this respect, the objects of mathematics 'exist'. So, mathematics is about 'existent' things (ὄντα). The Aristotelian principle mentioned above is thus verified.

Mathematics is (according to ARISTOTLE) primarily about the objects of the sense-perceivable world. But it deals with them only in a special way, namely, with regard to certain affections (παθήματα), such as 'indivisibility' (in arithmetic), or 'point form', 'line form', 'surface form', etc. (in geometry). The sense-perceivable objects themselves are therefore not examined *as such* in mathematics, but only *as* units, or *as* points, *as* lines, *as* surfaces, etc. (cf. *'Metaphysics'*, 1002b14-19).

In mathematics, however, one does not deal with the material carriers of units, lines, circles, surfaces, etc., but rather detaches oneself from them, so to speak, in order to be able to deal freely with the units, lines, etc., themselves (at least when they are discussed). Numbers and geometric objects are (in one sense or another) 'exempted' from the sense-perceivable world. They are perhaps not 'exempted' in the way the Platonic philosophers taught (namely, as objects related to 'ideas' and of which the worldly points, lines, curves and surfaces, etc., are only images), but ARISTOTLE is also convinced that they can be regarded somehow as 'separated' from the sense-perceptible things. In order to be able to explain this, ARISTOTLE additionally uses the term '*Chôrismos*'.

3.5. Chôrismós (Χωρισμός)

Chôrismos (the separation) describes the process of separating a mathematical object from its material carrier. The separation, however, can be executed in different ways. Therefore, the type of separation is the problem. Are mathematical objects absolutely separable or only separable in our thoughts? ARISTOTLE distinguishes these two alternatives with the following choice of words:

(†) χωριστὸν ἁπλῶς is a separated thing that exists autonomously.
(cf. *'Metaphysics',* 1042a31)
(ἁπλῶς = unconditional, plainly).

(‡) Λόγῳ χωριστόν is a thing that is treated like a separated thing only in speaking or thinking about it, but is not really separated.
(cf. *'Metaphysics',* 1042a29 and *'De anima',* 432a20 & 433b24-25).

For the Platonic philosophers, mathematical objects are separated from their sense-perceptible images, and therefore 'unconditionally separated' from them, as in (†).

According to ARISTOTLE, however, the mathematical properties are all bound to their material carriers and cannot be absolutely detached from them. But they can be detached in thought. In this 'detachment' (or 'separation'), no new (abstract) object is created, because the separation only takes place in our thoughts or in our speech.

For ARISTOTLE, the mathematical objects are λόγῳ χωριστά (as in (‡)). However, the mathematical objects are not really separated from their material carriers. Only thinking, when directed at them, can deal with them *as if they were independent things.*

In the 2nd book of the *'Physics*' (II, 193b31-36), ARISTOTLE describes this process of 'separation' as follows:

> *"Physicists, astronomers, and mathematicians have to deal with lines, figures, etc. But the mathematicians are concerned with them not bearing in mind that they are boundaries of natural bodies, and also not bearing in mind that their properties are manifested in such bodies. Therefore they separate (detach) the lines, figures etc from their material carriers; for they are capable of being considered in the mind in separation from the motions of the bodies to which they pertain, and such separation does not affect the validity of the reasoning or lead to any false conclusions."*

Just as, for example, the colour 'red' cannot exist by itself, but can only exist in material objects and has a fundament of its reality in these objects, so, according to ARISTOTLE, mathematical objects also exist only together with their material carriers and have their fundament of Being in them. They cannot be separated unconditionally from their material carriers. They do not exist outside of themselves, but need a material carrier for their existence. However, in the course of a mathematical investigation, they may be treated as if they were separated from their material carriers. ARISTOTLE writes, in the *'Metaphysics'* (XIII, 3, 1078a30)

> διττὸν γὰρ τὸ ὄν, τὸ μὲν ἐντελεχείᾳ τὸ δ' ὑλικῶς.
> [For Being is of two different kinds, either as independent reality, or as matter.]

The word 'entelechia' (ἐντελέχεια), which appears here, is composed of ἐν (in), τέλος (aim, destination, goal) and the verb ἔχειν (to have). It has the literal meaning of: 'that which has its goal within itself', and here, it means something like 'out of itself'.

Thus, according to the Aristotelian view, mathematical objects do not 'really' exist (i.e., they are not independently existing substances, as the Platonic philosophers taught), but only *in the manner of Hyle* (*hylikôs*, ὑλικῶς, i.e., materially). The mathematical objects therefore only exist in the same way as their material carriers, - cf. ARISTOTLE, 3rd book of *'Metaphysics',* 1001b1-1002b11.

In the process of 'detachment' (or 'separation'), no new object is created. The separation does not lead to independently existing objects, but only to a changed way of speaking.

3.6 The foundation of Arithmetic according to ARISTOTLE

According to the Aristotelian requirements, in constructing Arithmetic as a scientific theory, the first thing to do is to indicate the realm of objects that is to be investigated, and under which point of view the investigation will take place.

In his *'Second Analytics'* (72a23-24), ARISTOTLE says that, in Arithmetic, the objects of the real world are considered *as units*, and that all other properties are ignored. In particular, it is disregarded that they are (physically seen as) possibly divisible. In the *'Metaphysics'* (Book XIII, 3, 1077b27-34), we are told that, in Arithmetic, nature is examined *qua* indivisibility. In Arithmetic, then, multitudes of objects of the real world are considered, but only with respect to their number, whereby the individual objects themselves are regarded *as* 'units' (and thus 'indivisible' in the respective context).[8]

In arithmetic, therefore, according to ARISTOTLE, only one single genus of things is investigated, namely, the genus 𝔈 of all individual things in the world. All things of this genus are treated in Arithmetic as if they were generic objects of 𝔈, and are therefore regarded *as* units.

This having been said, the principles of Arithmetic must be given. The 'general principles' are the usual logical principles and the general scientific principles. They do not have to be stated again. Then, the 'special principles' (the *theses*, θέσεις) must be given. Since '*to be one thing*' can be said of all things individually (*'Metaphysics'* X, 2, 1053b20), the concept of unity does not need to be defined. In a hypothesis, however, the existence of units must be formulated (*'Second Analytics'*, 71a15-17 and 72a24).

On this basis, Arithmetic can be constructed. In particular, the *concept of the number* can now be introduced. ARISTOTLE defines it as follows. In the *'Physics'* (III 7, 207b8), he writes

(∗) *"Number is a multitude of units"* (πλῆθος μονάδων),

and in *'Metaphysics'* XIII, 9, 1085b22):

(∗∗) *"Number is a multitude of indivisible things"* (πλῆθος ἀδιαιρέτων).

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[8] In Chapter 2, we have seen that, according to PLATO, in scientifically operated arithmetic, the numbers are by no means multitudes of things of the natural real world, but multitudes of units that are obtained by abstraction, are all completely equal among themselves and are only accessible through thinking.

A few lines later in *'Metaphysics'* (XIII, 9, 1085b33-34), it says: "*Number is composed of indivisible things".* Numbers are thus multitudes of sense-perceivable things, which are to be regarded exclusively under the aspect of indivisibility (and thus *as* units).

A multitude of sense-perceivable things is usually not itself a sense-perceivable thing. The word 'multitude' is used here only in the colloquial sense, and not as a terminus technicus. Here, 'multitudes' always have at least two elements and are not things themselves (unlike in modern set theory, cf. Chapter 15).

Since 'units' are not 'multitudes', the unit is not a number. As ARISTOTLE writes:

> οὐκ ἔστι τὸ ἓν ἀριθμός. [The one is not a number],
> (*'Metaphysics'*, XIV, 1, 1088a6-7.)

The *numbers* 'two', 'three', 'four', ... can now be introduced by agreeing that the use of the word '*two*' should only mean: *"one unit and another unit, but no further unit"*, and similarly for '*three*', etc. ARISTOTLE calls these terms '*two*', '*three*', '*four*', ... '*numbers*' and writes (in his *'Physics',* book III, Chapter VII, 207b9) that they are only *'paronyms'*, i.e., words (nouns) derived from other words (adjectives).

The question of whether the numbers '*two*', '*three*', '*four*', ... can exist unto themselves, or whether the '*two*' can always be observed only in *two* things, the '*three*' always only in *three* things, etc., was often asked and discussed in antiquity. ARISTOTLE answered the question as follows: the individual units can exist unto themselves (they are *existing* objects), but the numbers '*two*', '*three*', '*four*', ... are only linguistic constructs (and not existing objects). [9]

But how are *addition*, *multiplication*, etc., introduced? ARISTOTLE does not explicitly deal with arithmetic operations (addition, multiplication) - see, however, *'Metaphysics'* XIII, 7, 1081b14 -, but it seems to be clear to ARISTOTLE that *addition* is not an operation that needs to be defined. It is an operation that is familiar and known to all people from everyday life as assembling (or adding) things from our environment. It is an *operation* that (according to all mathematicians from antiquity to the early 19th century) does not need to be addressed in mathematics, since it is well known and well understood outside of mathematics.

ARISTOTLE takes a standpoint in Arithmetic that is related to *nominalism.* Only the units have an existence; the numbers themselves are linguistic constructs. However, such a viewpoint makes the foundation of Arithmetic very difficult, and thus one would not get very far. We could hardly do more than set up addition- and multiplication-tables here.

3.7 The foundation of geometry according to ARISTOTLE

(1) The objects of geometry

From the description of the Aristotelian concept of a scientific theory, it follows that geometry studies the objects of the real world with respect to their shapes or forms.

[9] PLATO speaks about this question in *'Hippias maior',* 301b-302d. Additionally, PLOTINUS, the founder of Neo-Platonism, dealt with it extensively and very subtly in his treatise *'Of the Numbers'*: Περὶ ἀριθμῶν, 9,6.

Geometry is, hence, a *science of quantity*.

In constructing geometry as a scientific theory, the first thing to do is to introduce all species or genera of objects that should be studied. In the first place, the genus of things that have *length* must be introduced, and it has to be declared that its objects are to be treated as if they were generic objects of their genus and that they should be regarded *as straight lines without breadth.* However, in order to obtain a scientific theory of geometry, the concept of 'straightness' (or of 'length') has to be introduced by a definition in which the essence of that concept is clearly defined. What is the *causa formalis* for '*straightness*'? ARISTOTLE gives no answer. What is known is only a device for constructing straight lines by using mechanical tools.

The introduction and naming of the other species or genera of geometric objects poses similar problems. What, e.g., is the *causa formalis* for a *plane superficies*?

ARISTOTLE discusses, in his *'Topics'*, the definitions that were common at that time: *"The point is the boundary of a line"* (141b6-22), *"The line is the boundary of a surface"*, and *"The line is length without breadth"* (*'Topics'*, VI, 143b13-33), etc. He criticized these definitions because they do not describe the essence of objects comprehensively, but failed to give any better definitions himself.

(2) The principles of geometry

The *general principles* are the *general scientific principles*, as well as the usual *generally valid logical statements* and the proof-theoretical rules, as far as they were known at that time. They need not be listed again (see above).

The *specific principles* are the *theses*, i.e., the *definitions* and the *hypotheses.* The basic concepts have to be defined through definitions of their essence. Then, the definitions of the derived technical terms (such as 'triangle' (cf. *'Second Analysis'* I,1; 71a14-15), 'acute-angled triangle', etc.) can be formulated as usual.

ARISTOTLE is largely silent about the hypotheses underlying geometry. It is required, however, that the pure existential propositions *"There are points", "There are straight lines", "There are circles"*, etc., appear among the hypotheses (cf. *'Second Analytics'* I,10; 76a34-36; 76b6). This means that the species and the genera of geometric objects are not empty (cf. *'Second Analytics',* 76b18) and that, consequently, all 24 syllogisms derived by ARISTOTLE in his *'First Analytics'* are available without restriction. [10]

As described by ARISTOTLE in his *'Second Analytics'*, further hypotheses can be established, provided that they are 'true' (cf. *'Second Analytics'* I,2; 71b26). These hypotheses can be directly insightful statements or statements that can be obtained through induction (ἐπαγωγή, see above) from directly insightful statements (cf. *'Second Analytics'* I,2; 99b15-100b5).

This is, in brief, an outline of the foundations of geometry in the sense of ARISTOTLE, and one could begin with the proofs of the various theorems.

As soon as a diagram is presented with the help of rulers, right angles, compasses

[10] For this purpose, syllogisms must be formulated in a language with several sorts of individual variables, whereby the variables of one sort always run through only one genus of objects - cf. TIMOTHY SMILEY *'Syllogism and Quantification'*, J. of Symbolic Logic 27 (1962), pp. 58-72.

(possibly even generalized compasses for drawing conic sections), etc., the transition to the generic objects indicates that the drawing is to be understood as a sense-perceptible representation of what can only be seen (due to the underlying definitions of the essence of the individual types of objects) with the eyes of the mind, and that the following proof refers only to this mentally perceptible configuration and, hence, is also exact.

For example, a sense-perceivable circular line on a piece of paper may be thicker or thinner in some places and may be a little angular and inaccurate in some places. But for use in mathematics, the line is a 'circle' exactly when it is known that it has been obtained (or: generated) according to a certain rule (e.g., by using a compass), and therefore belongs to the species of circular objects. The properties resulting from the underlying rules are relevant for their uses in mathematics, and not the empirically ascertainable appearances.

ARISTOTLE wrote, in his *'Second Analytics'*, I,X (76b39-77a1):

> Οὐδ' ὁ γεωμέτρης ψευδῆ ὑποτίθεται, ... τὸν δὲ γεωμέτρην ψεύδεσθαι λέγοντα ποδιαίαν τὴν οὐ ποδιαίαν ἢ εὐθεῖαν τὴν γεγραμμένην οὐκ εὐθεῖαν οὖσαν. ὁ δὲ γεωμέτρης οὐδὲν συμπεραίνεται τῷ τήνδε εἶναι γραμμὴν ἣν αὐτὸς ἔφθεγκται, ἀλλὰ τὰ διὰ τούτων δηλούμενα.
>
> [The geometrician's hypotheses are not false, ... (when) he asserts that the drawn line is a foot long, or straight, although it is not straight. A geometrician does not draw any conclusions from the appearance of an existing, concrete line he is talking about, but only from what is made clear by it.]

Here, it is expressed very nicely that the proofs of geometry are not about what is sense-perceptible, but about the concepts underlying the geometric figures that are mentally perceptible, and that it is not sufficient to perceive the diagrams only with the senses, but that it is necessary to recognize that they were created by applying certain rules!

However, geometry as a scientific discipline based on Aristotelian philosophy cannot be adequately carried out in all respects, because the existence of geometric objects is dependent on physical conditions. The usual proofs of the geometric theorems can only be carried out in Aristotelian geometry *if all the auxiliary lines required in the proof are also present and if these auxiliary lines are unambiguously determined, when necessary.*

The following theorem, mentioned by ARISTOTLE

> **Theorem** *of the sum of the interior angles of a triangle* (*'Metaphysics'* IX, 9, 1051a22-26, and *'Eudemic Ethics'*, II,6, 1222b23-41),

needs, for its proof, an auxiliary straight line, which goes through a corner of the triangle and is parallel to the opposite side of the triangle. It is unclear whether such an auxiliary line even exists for each triangle in the Aristotelian-conceived geometry and, if it does exist, whether it is unambiguously determined and unique. Whenever such a parallel line exists and is uniquely determined, the usual proof (as in the *'Elements'* of EUCLID, Book I, § 32) can also be carried out correctly in Aristotelian geometry. But if this parallel does not already exist together with the given triangle, then there is no hypothesis in Aristotelian geometry that guarantees its existence and unambiguity and the proof is not feasible!

This is the central problem of Aristotelian geometry: it does not have the *postulates* that are present in Euclidean geometry. The existence of auxiliary lines in Aristotelian geometry

is only assured if they can be obtained using the method of abstraction (*Aphairesis*). For this purpose, the auxiliary lines must be present *in the manner of Hyle, i*.e., 'material', so that they can be obtained by means of *aphairesis* and *chôrismos* as objects of geometry. SIMPLICIUS (Σιμπλίκιος) had already noticed (according to the testimony of ANARITIUS, that is, *Al-Nayrizi*) that it is questionable whether it is possible to produce the necessary auxiliary lines everywhere on earth (for example, on the curved surface of the sea) and in space. And even if one could produce them, it is questionable whether they are uniquely determined, as the Euclidean parallel postulate demands.

In Aristotelian geometry, the provability of geometric theorems obviously depends on physical conditions of the natural real world. But geometry cannot make any statements about this. This makes the Aristotelian-conceived geometry ultimately unfeasible (cf. Chapter 4).

As much as ARISTOTLE initiated and promoted the development of logic and the exact sciences, and, in doing so, brought to light a great deal of profound knowledge about the foundations of mathematics, he also linked mathematics far too closely to the natural real world. In his thesis that the objects of mathematics are things that are obtained through abstraction (τὰ ἐξ ἀφαιρέσεως) from objects of the real world, he subordinated mathematics to the natural sciences, and thus did not adequately estimate questions concerning the ontology and methodology of mathematics.

For this reason, further development of mathematics on the grounds of Aristotelian philosophy is hardly possible. The negative numbers, the complex numbers, the infinite-dimensional HILBERT-*spaces*, etc., none can be obtained through the processes of abstraction and separation (*aphairesis* and *chorismos*) from objects existing in the real world. Again, the all-too-close connection to the empirically perceivable world does not allow for the development of mathematical thinking.

In the rather long history of geometry, it ultimately became clear that the subject matter of geometry is not the real world. *Geometry cannot investigate things in the real world, - geometry only refers to a framework of concepts.* According to our modern view of mathematics, geometry, as a scientific theory, is the study of models of such frameworks, and not the study of the real world (cf. Chapters 4, 19 & 20).

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Chapter 4
The Axiomatic Method of EUCLID

For the Babylonians and Egyptians, as well as for the Greeks in pre-Pythagorean times, geometric objects were the drawn straight lines and circles, their intersections and the areas enclosed by the drawn lines. In astronomy, points, lines and areas were also considered, but they were thought to be 'drawn' into the vault of heaven. The breadth of all these lines, whether drawn or only imagined, was not considered, but it was also not assumed that they had only length and no breadth at all. Further, it was not assumed that the points were without extension - otherwise, they would not have been perceptible at all.

Just as the existence of all these geometric quantities is already secured by their visibility, the geometrical statements that refer to such quantities are statements that can also be verified by ocular demonstration.

We have seen in Chapter 1 that the discovery of the incommensurability of the side and diagonal of a square (around 470/450 BCE) led to the explicit assumption that, in the new theoretical geometry, points are extensionless positions and lines are breadthless lengths. Such points and lines are no longer perceptible to the senses, and it is therefore unclear in what sense they exist and from what sources we can draw when we want to prove statements about configurations made out of such objects. (We already asked this question in the introduction.)

In Chapters 2 and 3, we have discussed the attempts by PLATO and ARISTOTLE to interpret and clarify the ontological status of mathematical objects and the epistemological status of mathematical theorems and theories. Their attempts were deeply thought out, but still could not fully convince the mathematicians.

Mathematicians have taken a path in the development of arithmetic and geometry that should be largely independent of ideological positions. This path was that of *axiomatics.* EUCLID worked out such an axiomatic treatment of geometry and of arithmetic in his *'Elements'*, and we shall report on this in the following.

Almost nothing is known about the life of EUCLID. His place of birth and ethnic origin are unknown. We only know that he was probably born around 340 B.C.E., and died around 270 B.C.E. (cf. JOHANN LUDWIG HEIBERG 1882, op. cit., pp. 25-26), that he lived in Athens for some time and that he was appointed to the Museion in Alexandria by PTOLEMY I around 300 B.C.E. The Museion was a sanctuary of the Muses, having the function of a research institute and also housing a large library. When EUCLID moved from Athens to Alexandria, Athens lost its leading role in mathematics.

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EUCLID wrote many works on pure and applied mathematics. His magnum opus is the thirteen books of the *'Elements'* (Στοιχεῖα), which he probably wrote in the years around 300 BCE.[1] PROCLUS (Πρόκλος, ca. 411- 485) reported that EUCLID summarized many results of EUDOXUS (Εὔδοξος, ca. 400-347 B.C.E.) in the *'Elements'*, brought many results of THEAITETUS (Θεαίτητος, 415/413-369 B.C.E.) to a conclusion and put the less stringent proofs of his predecessors into irrefutable form.

The well-organized structure and architecture of this thirteen-volume compendium was of particular significance. The strict, logical structure became proverbial for an exact representation: *'more geometrico'* (i.e., in the manner of EUCLID′s geometry).

4.1 The *'Elements'* (Στοιχεῖα) of EUCLID

In the *'Elements'*, EUCLID treated geometry and arithmetic. At the beginning of the presentation of geometry, all terms of fundamental importance are mentioned and explained, as PLATO and ARISTOTLE had requested. In addition, EUCLID gives a list of postulates and a list of general principles relating to the concepts of equality and size. This is what EUCLID's version of this looks like:

Definitions (ὅροι, lat.: *definitiones*):

1. *A point* (σημεῖον, *signum*) *is that which has no parts,*
2. *A line* (γραμμή) *is length without breadth,.*
3. *The extremities* (πέρατα) *of a line are points,*
4. *A straight line* (εὐθεῖα γραμμή, *linea recta*) *is a line which lies evenly* (ἐξ ἴσου, *ex aequo*) *between its [extreme] points,*

etc. The concepts of *area, angle, circle, diameter, parallelism*, etc., are then defined. The concept of a circle is defined as follows:

15. *A circle (κύκλος, circulus) is a plane figure contained by one line [which is called the circumference] such that all straight lines falling upon it from one point among those lying within the figure are equal to one another.*

Postulates (αἰτήματα, lat.: *postulata*, French: demandes): *It is postulated,*

1. *that a straight line may be drawn from any one point to anyother point,*
2. *that every straight line of limited length can be continuously extended* [as *far as one likes*] *in a straight line,*[2]

---

[1] ARISTOTLE wrote, in his *Metaphysics* (Δ.3. 1014a37-1014b3), that the proofs that recur as components in other proofs were called *'elements of proofs'.* This may explain the title of EUCLID*′s* book.

[2] In the Greek text, the word ἐκβαλεῖν (to extend, to throw out, to throw away) is used. It is remarkable that, in some early Latin translations, the second postulate is formulated with the addition *"quantumlibet protrahere*" (e.g., in version 2 which ADELARD of Bath made from Arabic around 1120), or *"quantumlibet protrahatur"* (e.g., in the Lüneburg manuscript from the second half of the 12th century, which goes back to BOËTHIUS) - compare MENSO FOLKERTS, op. cit. However, this addition is not found in the commentary by PROCLUS (op. cit., p. 296) and also not in the Latin translation, which was made around 1160 in Palermo directly from a Greek manuscript from Byzantium - cf.BUSARD, op. cit., p. 28.

Καὶ πεπερασμένην εὐθεῖαν κατὰ τὸ συνεχὲς ἐπ' εὐθείας ἐκβαλεῖν,

3. *that for any two points A and B a circle with centre A may be described such that B is on its circumference,*
   Καὶ παντὶ κέντρῳ καὶ διαστήματι κύκλον γράφεσθαι,
4. *That all right angles are equal to one another,*
5 [*the parallel postulate*] (see Chapters 5 and 8).

Common notions (κοιναὶ ἔννοιαι, lat.: *communes notiones*):

1. *Two things which are equal to a third thing are also equal to one another.*
2. *If equals be added to equals, the wholes are equal.*
3. *If equals be subtracted from equals, the remainders are equal.*
   ... etc. ...
7 *Things which coincide with one another are equal to one another.*
8. *The whole is greater than its part.*
9. *Two straight lines cannot enclose a space.*

In the following, we shall carefully examine some of the details in EUCLID′s text.

4.2 The terminology in the *'Elements' of* EUCLID

EUCLID does not use the old familiar terminology. For all older geometers, *'stigma'* (στίγμα, στιγμή) is the technical term for *'point'*. It has the original meaning of *'stitch',* a small hole created by pricking (στίζω), just like the Latin word *'punctum'*, which is derived from the verb *'pungere'* (to prick). *'Center'* (κέντρον) is the *'mark'*, which is pricked with a *'sting'* or *'spine',* or with a compass.[3] The word κέντρον is also represented in Latin with *'punctum'*, as CICERO reports in his '*Tusculanae Disputationes*', I,40. The words *'stigma'* and *'kentron'* thus refer to something that is sense-perceptible. EUCLID, however, avoids such words.

In the *'Elements'* of EUCLID *'semeion'* (σημεῖον, lat. *signum*) is the technical term for 'point'. This word means *'sign', 'mark', 'mark of distinction'*. Since EUCLID′s geometry refers to *imagined* (or *ideal*) objects, the places where two straight lines or two circles intersect, and the places where lines begin and end are not referred to as 'punctures'. For such places, the more abstract word 'semeion' (marking) is certainly more appropriate than the word 'stigma' (puncture).

Most Greek mathematicians adopted the Euclidean terminology, in particular, HERON of Alexandria (he probably lived around the year 62). AURELIUS AUGUSTINUS (354-430) emphatically defended EUCLID′*s* choice of words in his dialogue *'De quantitate animae'*

[3] Notice that, in Algebra, the *'center'* of a group is not the midpoint of a group *G* (what should that mean?), but consists of all *'central'* elements, i.e., those elements *g* whose conjugacy class is a singleton (equivalently, those elements *g* that remain fixed under all inner automorphisms). These elements had been called *'isolated'* elements by HEINRICH WEBER in his *'Lehrbuch der Algebra'* (1899), vol. 2, p. 133. J.-A. DE SÉGUIER, in his book *'Théorie des groupes finis - Élements de la Théorie des groupes abstraits',* Paris 1904, p. 57, called these elements *'central'*, since their conjugacy class is just a *single point,* i.e., a *'punctum'* or *κέντρον*.

(XI.18). He translated them into Latin and named the starting and end points of lines and the intersections of curves *'signa'*. Only for the centre of a circle did he use the term *punctum* (just like CICERO).

But EUCLID, HERON, AUGUSTINE and others did not prevail in their terminology, because the word στίγμα (*punctum*) seems to suggest a greater reality than the word *semeion* (*signum*). In Latin literature, the translation *'punctum'* had already been established in the classical period. It is therefore not surprising that the word *'punctum'* has become common in all modern European languages as well. But, nevertheless, one must explicitly point out again and again that 'points' in geometry are not stings, pricks or punctures, and are also not sense-perceptible. SIMON JACOB does this very aptly in his arithmetic-book from 1565:

> *"Ein Punct ist ein untheilbares reines stüpflein, welches mit keinem Instrument mag gemacht, sondern muß allein mit dem verstandt gefaßt werden."*
>
> [A point is an indivisible, rather small dot, which cannot be made with any instrument, but must be grasped with the mind alone.]
>
> SIMON JACOB: *'Ein New und Wolgegründt Rechenbuch, auff den Linien un Ziffern, ...'*, Franckfurt am Mayn, 1565.

It is remarkable that EUCLID never speaks of *'compass'* (κίρκος, κρίκος) or *'ruler'* (κανών), but always only of *'circle'* (κύκλος) and *'straight line'* (εὐθεῖα γραμμή). As a result, mechanically executable constructions are not the subject of the *'Elements'*, but only conceptually given objects. The question is how these objects are introduced. How are they defined? Is the term *'straight line'* defined correctly?

4.3 What should the *'definitions'* achieve?

EUCLID begins the first book of *'Elements'* with a list of 23 definitions. Some definitions are *nominal definitions* and others *definitions of concepts*. The *nominal definitions* are unproblematic. But the first seven definitions, in which the basic concepts of *point, line, straight line, plane* and *flat surface* are introduced, are not nominal definitions. They raise big problems, and we have to analyse and interpret them.

The translation of the first definition into English states that a *point* is something *'which has no parts'*. A *'point'* is thus nothing more than an *'individuum'* in the literal sense of the Latin word (i.e., an *indivisible* thing). It is remarkable, however, that no attempt is made to capture the descriptive content, but only the formal property of indivisibility. Among the geometric objects, points are the only ones that cannot be divided into parts. Indivisibility is thus a characteristic property of points in Euclidean geometry. It is remarkable, however, that EUCLID does not aim for a description of the intuitive content of what a point is.

The second definition states that *lines* (straight lines and circles) are those things that have length but no breadth. This definition is based on the view that has been developed in geometry since the discovery of incommensurable quantities (probably by HIPPASUS) (see Chapter 1). Although this definition communicates an essential property of lines, it does not fully define the concept of a line, since it does not say what 'length' and 'breadth' are.

The fourth definition describes the 'idea' of a *straight line.* The formulation is not easy

to understand (see section 4.1, def. 4, for an English translation):

Εὐθεῖα γραμμή ἐστιν, ἥτις ἐξ ἐξ ἴσου τοῖς ἐφ' ἑαυτῆς σημείοις κεῖται.

CRISTOPH CLAVIUS (1538-1612) gave the following Latin translation in his edition of EUCLID*'s 'Elements'* (3rd edition, Cologne 1591):

Recta linea est, quae ex aequo sua interiacet puncta.

A 'straight line' is therefore a line that is *even* or *uniform* (ἐξ ἴσου, *ex aequo*) between 'its' points. It is probably only 'its' *end points* that are being referred to, as in the third definition. A 'straight line' therefore seems to be a line that extends from one end point to the other end point without changing its direction. But it is not quite clear whether EUCLID means to say this. What he says are not definitions of concepts but only allusions to mental representations of basic geometric terms. Is it at all possible to define the term 'straight line' in simple geometric terms without the help of visualization?

Definitions 5, 6 and 7 refer to *surfaces*, the *extremities of surfaces* and *plane surfaces*. These are transfers of the corresponding definitions 2, 3 and 4, and can therefore be skipped here. But it must be stressed that the definition of the term *plane surface* (definition 7) is as problematic as the definition of the *straight line* in definition 4.

It has become clear that the first seven definitions are completely different in nature from the rest: they are neither nominal definitions nor definitions of concepts. But, perhaps they do communicate (in the spirit of a Platonistically-conceived geometry) some of the essential characteristics of the geometric objects in question, and so they are indispensable in the construction of Euclidean geometry. Here, the *'nous'* should be able to recognize the corresponding ideas.

However, the question still arises as to whether these first seven definitions were written by EUCLID at all, or whether they were interpolated by later editors of the *'Elements'*. It was already conjectured by JOHANN LUDWIG HEIBERG in 1882 (op. cit., p. 193-194) that they were only inserted into the *'Elements'* by HERON (who lived in the second half of the 1st century) - see also LUCIO RUSSO 1998, op. cit.

4.4 What should the *'common notions'* achieve?

EUCLID speaks of κοιναὶ ἔννοιαι, which, nearly faithfully translated, means *'knowledge which is common to everyone'*. The word ἔννοια is derived from the verb νοεῖν (which means: 'to notice', 'to look through', 'to know' and ἔννοεῖν 'to think over', 'to imagine', 'to reflect about', 'to bear in mind'). Thus, ἔννοια is the knowledge obtained through introspection alone, and hence is common to all of us (cf. Chapter 2).
In contrast to the postulates whose acceptance must be demanded or requested, since they are not universally valid, the κοιναὶ ἔννοιαι are statements whose validity hardly anyone will dispute.

The expression κοιναὶ ἔννοιαι has different translations in Latin. With A.M.T.S. BOETHIUS (circa 505), ADELARD of Bath (around 1120) and CAMPANUS (circa 1255/1259), we read *'communes animi conceptiones'*, and with CHRISTOPH CLAVIUS (1574), TH.L. HEATH (1956) and others, *'communes notiones'*. The translation *'axiomata'* can also be

found (PROKLUS, G.A. BORELLI (1679), ISAAK BARROW (1685) and others). However, the translation *'communes notiones'* seems to be the most accurate. Notice that *'notio'* in Latin (or *'notion'* in English) does not mean *'concept'*, but rather *'thought'* (see Chap. 2, section 2.5).

Under the heading κοιναὶ ἔννοιαι (common notions), EUCLID mentions some statements concerning *equality* (however, not concerning identity!). Most of them are quite obvious. But when are two angles equal? Such a definition is missing in the *'Elements'* of EUCLID.

It is stated in the 7th common notion that *things which coincide with one another are equal to one another*: καὶ τὰ ἐφαρμόζοντα ἐπ᾽ ἄλληλα ἴσα ἀλλήλοις ἐστίν. The word *epharmózein* used here (ἐφαρμόζειν, Lat.; *congruere,* to be congruent) raises problems, because it is not explained how it can be used in theoretical geometry without reference to sense-perception.

ÁRPÁD SZABÓ (op. cit., p. 359) assumed that the 8th common notion (*'The whole is greater than the part'*) was established to avoid annoying discussions that might arise with the followers of ZENO. ZENO (born about 490 B.C.E. in Elea), for example, had established and defended the paradox that half the time is equal to twice the time (cf. ARISTOTLE: *'Physics Lecture'*, Book VI, 240a1). In the *'Nomoi'* (Book III, 690 e), PLATO reminds the guest from Athens that HESIOD *'very correctly'* claimed *'that half* is *often more than the whole'*.

The 9th common notion *('Two straight lines cannot enclose a space')* was probably introduced later to support the proof of Theorem **I**.4. It does not belong to the category of the 'common notions', as PROCLUS already stated in his commentary on the *'Elements'* (op. cit., p. 302 & p. 333-334). One could omit the 9th 'common notion', if one were to somewhat tighten the first postulate as follows:

1*. It is postulated *that one and only one straight line may be drawn from any one point to any other point,*

4.5 What should the *'postulates'* (αἰτήματα) achieve?

Aitêma (αἴτημα), in Greek, is something that is *'requested', 'required', 'demanded', 'postulated'*. The Latin translation of the word is *postulatum,* and the English translation is *'requirement'* or *'postulate'*. In the first book of the EUCLID*ean 'Elements'*, we are told that we are *required to accept* the 5 statements, called 'postulates', not as 'true' statements, but as a basis for the further development of plane geometry.

The postulates require that one be able to 'draw' (more precisely: 'to bring on', 'to produce', ἄγειν) certain lines, and that one accepts that all right angles are equal to each other, along with a certain statement about parallel lines. EUCLID thus demands of the person who wants to engage in geometry that they accept the postulates stated as fundamental assertions in full generality.

EUCLID does not demand that one be able to draw a circle in the real world around any point with any radius, nor does he demand that one be able to draw a line from any point

to any other point with a suitably long ruler (γράφειν). Such requirements would certainly not always be feasible. It is not possible, for example, to draw a line from the earth to the moon, nor to produce straight lines on the curved surface of the sea, as SIMPLICIUS (Σιμπλίκιος) had already noted (cf. Chapter 3). EUCLID does not speak of the (sense-perceptible) 'drawing' of lines, but only, quite indefinitely, of 'bringing them on'.

The postulates do not refer to the use of mechanical tools (such as rulers and compasses), as already mentioned. EUCLID does not write about these tools anywhere in his *'Elements'*. The postulates rather demand the readiness to deal with the given *terms* (!) in the given way.

The EUCLIDean postulates therefore demand that, in thoughts, one always be able to pass from two *imagined* points to the *imagined* connecting straight line and that one always be able to pass from two *imagined* points to the *imagined* circle line, which has the first point as its centre and passes through the second point, etc.

EUCLID's postulates are statements that refer to a world of imagined and idealised objects. They demand that the basic constructions that could be carried out with mechanical devices (e.g., connecting two points with a straight line by the use of a ruler, extending straight lines, parallel displacements of straight lines, describing circles, etc.) should also be available in the new geometry of imagined and idealised objects.

In particular, the ontology of geometric objects proposed by ARISTOTLE (cf. Chapter 3) is not adopted. *The Euclidean postulates rather allow us to close the gaps that cannot be closed in the ontology of geometrical objects as represented by* ARISTOTLE (cf. Chapter 3, section 3.7). It is remarkable, however, that, in the *'Elements'* of EUCLID, the ontological status of geometric objects is not discussed anywhere.

EUCLID himself is very reticent about such explanations. But, in a few places, he writes explicitly of 'imagined' circles and 'imagined' lines. In the 12th book, for example, problem 17 is opened with the following words:

> Νενοήσθωσαν δύο σφαῖραι περὶ τὸ αὐτὸ κέντρον τὸ Α.
> [*Fingantur duae sphaerae circum idem centrum A.*
> Let us imagine two spheres around the same center *A.*]

Conclusion: It is evident now that EUCLID does not speak of sense-perceivable points and lines, etc., and that the constructions that are performed are not constructions created with mechanically executable tools such as ruler and compass. The geometry that is developed in the *'Elements'* deals with objects that are *idealizations and abstractions* of perceptible real objects in such a way that it is possible to obtain propositions and theorems (in full generality) about *all* (!!!) objects of a certain kind: all triangles, all rectangles, etc. This was not possible in the old, pre-Pythagorean geometry of the Babylonians and the Egyptians on an empirical basis (cf. Chapter 1, end of section 1.1).

- It has long been customary also to call the *Euclidean* postulates *'axioms'*. Is such a designation permissible? What is the difference between an *'axiom'* and a *'postulate'*?

4.6 *Axioms, postulates, hypotheses* and *lambanomena*

In scientific studies, as well as in dialectical disputes, it was already common practice in

antiquity to explicitly name the statements upon which one wanted to base oneself, but which one did not want to (or could not) justify. Such statements were called *'axioms'* (ἀξιώματα), *'lambanomena'* (λαμβανόμενα), *'hypotheses'* (ὑποθέσεις) or, occasionally, also *'postulates'* (αἰτήματα). What is the original meaning of these words?

1. Axioms. The etymological dictionaries show that, in the Ionic and Attic dialect, *axia* (ἀξία) meant *'value', 'reward'*, from which the words *axiôsis* (ἀξίωσις), i.e., *'appreciation of value'*, *'reverence'*, *'prestige', 'good opinion'*, *'pretension',* and *axiômatikos* (ἀξιωματικός), i.e., *'weighty', 'dignified', 'grave'*, were derived. The verb ἀξιοῦσθαι means *'to be of value', 'to be of great importance', 'to be held in great esteem'*.

The word *Axíôma* (ἀξίωμα) has the colloquial meaning of *'appreciation', 'dignity', 'validity', 'importance'*. In the great historical work of POLYBIUS, ἀξίωμα τοῦ βασιλέως means the *'dignity of the king'*. Notice, however, that *'validity'* is not the same as *'truth'*.

If a statement is called an *'axiom'*, then it is a statement that has validity, or at least one that claims validity (i.e., whose validity is postulated). This means that such a statement has a special *weight.* In mathematics, an *'axiom'* has the *dignity* and *weight* to be *among the first assertions of a theory.*

The word *'axiom'* was also used by ARISTOTLE in this sense. He had called the logically universally valid statements κοιναὶ δόξαι (*'Metaphysics'* III,2, 997a20-22), and, to emphasize their special importance, had also called them *'axioms'* (ἀξιώματα).

In the Stoa, the word *'axiom'* was given the slightly modified meaning of *'statement which is immediately understandable without further explanation'*. The Stoic CHRYSIPPUS (ca. 280-205 B.C.E) gave the following definition:

> *"An axiom is that which in itself can be used as a statement or proposition."*
> (quoted after MALTE HOSSENFELDER, op. cit. p. 241.)

AULUS GELLIUS (he lived around 130) gave another explanation. In his work entitled *'Noctes Atticae',* Lib. XVI, Cap. 8, he says that the word *'axiom'* (lat. *'pronunciatum'*) would designate an absolutely independent principle, which is explained only by itself, and therefore does not have to be proven. He quotes the Roman poet and scholar MARCUS TERENTIUS VARRO (116-27 B.C.E.), who rendered *'axioms'* in Latin with *'profata'* ('sayings') or even *'proloquia'* (*'proverbs')*, and is purported to have said that a *'proloquium'* (*'axiom'*) is understood to be an expression of opinion *"in which nothing is missing"* [4] (i.e., whose content is immediately clear):

> *"Proloquium est sententia, in qua nihil desideratur."*

AULUS GELLIUS gives some examples: *"Hannibal Poenus fuit"* (HANNIBAL was a Punic), *"neque bonum est voluptas neque malum"* (Enjoyment is neither good nor bad), etc.

From these quotations, we can see that, in antiquity, an *'axiom'* was understood to be a statement that was meaningful in itself, and which can therefore be placed at the beginning of a theory and be claimed as valid (cf. also the Epilogue, E.1).

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[4] *Profari* means, in Latin, "to proclaim, to predict, to prophesy", and *proloqui* "to pronounce, to proclaim, to prophesy". *Pronunciare* means "to proclaim, to announce".

2. Lambanomena (accepted statements). ARCHIMEDES, in his treatise on *'Sphere and Cylinder',* gave his list of definitions the heading *'Axioms'*, in order to indicate that they were the very first propositions of his theory. The subsequent list of hypotheses, the truth of which he considered to be evident but which he could not prove, he called *'lambanomena'* (λαμβανόμενα), i.e., *'adopted statements'*, in reference to the dialecticians. The word is derived from the verb λαμβάνειν (*lambanein* = to take, to accept, to assume, to adopt, to pose, to suppose, to grasp, ...).

3. The meaning of the term **'hypothesis'** was already discussed in detail in Chapter 3, section 3.1.

4. In the **postulates**, which EUCLID placed at the beginning of his *'Elements'*, it is demanded that the basic constructions that could be carried out in pre-Pythagorean geometry with mechanical devices (e.g., connecting two points by a straight line, extending straight lines, parallel displacements of lines, describing circles, etc.) should also be considered valid in the new geometry, which deals with points without extension, and lines without breadth, etc. These are not statements that are valid *a priori*, but rather statements whose validity must be postulated.

The Euclidean postulates are propositions that can only be understood when the terms occurring in them are known. That is why the 'postulates' are only mentioned after the 'definitions'. Thus, the Euclidean postulates are not *'axioms',* neither in the sense of ARISTOTLE nor in the sense of the Stoa. Since they only refer to the handling of idealized objects, they are neither true nor false. But they are propositions that claim validity in the context of geometry. They are propositions, which, together with the definitions, are the first assertions, or propositions, of geometry. Therefore, they can also be called 'axioms', if one likes. Since this use of language has become common, we choose to adopt it.

The Euclidean axiomatic theory is based on the following idea: first, *all sources* are mentioned from which one may draw if one wants to gain geometric insights. These sources consist of lists of *definitions*, lists of *postulates*, and lists of *common thoughts*. To draw from these sources means to draw conclusions from the statements contained in the lists through the means of pure logic.

The entire content of geometry is therefore contained in the three lists mentioned above.

- After we have discussed the basics in detail, let us take a brief look at how geometry itself is treated and how its theorems are drawn from the sources mentioned.

4.7 The representation of Geometry in the *'Elements'* of EUCLID

In antiquity, many geometric problems could be solved using only a compass (κίρκος) and a ruler (κανών). The fact that a drawing executed with these aids gave the solution to the posed problem could be perceived in most cases with the eyes. However, if one wanted to prove that there is indeed a solution, one had to free the argumentation from all *'Erdenrest' ('earthly residue')* and proceed purely conceptually. In the argumentation, the reference to the *tools* 'compass' and 'ruler' had to be replaced with the *concepts* 'circle' (κύκλος) and 'straight line' (εὐθεῖα γραμμή).

Such a demonstrative geometry was worked out by EUCLID in his *'Elements'*. The propositions there are either descriptions of mental constructions (these are the *problems,* προβλήματα) or theorems (these are the *theorems*, θεωρήματα). A solved problem ends with the abbreviation QOF (quod oportebat fieri, ὅπερ ἔδει ποιῆσαι, which should be executed) and a proved theorem with the abbreviation QED (quod erat demonstrandum, ὅπερ ἔδει δεῖξαι, which had to be proved). The Greek verb *deiknymi* (δείκνυμι) used here means *'to show', 'to let see', 'to make understand', 'to prove'*.

In the proofs of theorems, very often, additional lines or other configurations are used, and these must be introduced into the course of the proofs. For this purpose, EUCLID uses the constructions carried out in the 'problems'. Conversely, EUCLID uses the theorems in order to prove, that the proposed constructions do have the desired properties. Thus, the solved problems and the proved theorems are closely interwoven. Just as the theorems deal with imagined (or idealized) points, lines, surfaces, etc., so too do the problems deal with such imagined/idealized objects. The objects must meet the same definitions in the problems as in the theorems. This means, in particular, that the problems do not deal with constructions with the help of mechanical tools (such as compasses and rulers). - This, unfortunately, is overlooked quite often in the literature.

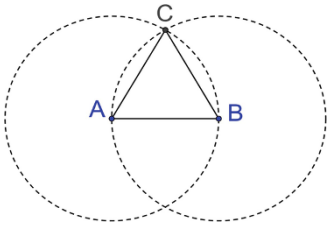
We will now take a brief look at some of the proofs in EUCLID*'s 'Elements'*. We begin with a look at the solutions to two problems and the proof of a theorem in the first book.

4.8 The arguments in the problems, resp. theorems I,1 and I,2 and I,4

Problem I,1: *On a given finite straight line to construct an equilateral triangle.*

Solution: Let *A* and *B* be the end points of the given straight line (definition 3). According to postulate 3, there is a circle around the center *A* with radius *AB*. Similarly, there is a circle with the center *B* and radius *BA*. Let *C* be one of the two points of intersection of the two circles. According to the definition of the term *'circle',* the lines *AB* and *AC* are of equal length.

Similarly, the two lines *BA* and *BC* are of equal length. The line with the end points *A* and *B* may be denoted by *AB*, as well as by *BA*, thus *AB* = *BA*. According to the first common notion, the lines *AC* and *BC* are also of the same length. So, the three lines *AB, AC* and *BC* all have the same length and, hence, the three points *A, B, C* define an equilateral triangle, QOF (quod oportebat fieri).



The description of the solution to the problem shows very clearly how the presupposed definitions, postulates and common notions are used. But it is also evident that EUCLID has

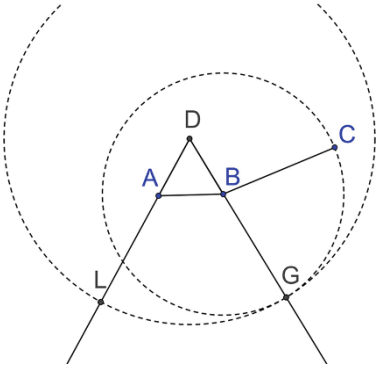
already been a bit careless in the very first problem. That the two circles intersect is not proved, but only taken from sense-perception. For a complete proof, one needs the insight that the two circle lines are continuous, and therefore intersect.

In HILBERT's axiom system (we will talk about this in Chapter 20), problem **I**,1 is solved as follows: First of all, it has to be shown that the straight line *AB* can be halved (compare DAVID HILBERT: *'Grundlagen der Geometrie'*, 10th edition 1968, §6, p. 25, theorem 26, but compare also EUCLID *'Elements'*, **I**,10) and that, consequently, the perpendicular line through the midpoint of *AB* can also be constructed. This perpendicular line intersects the two circles at a common point, which can easily be proved using DEDEKIND's axiom of continuity.

We look at the next problem **I**,2, in the *'Elements'* of EUCLID, because it is well suited to clearing up possible misunderstandings that may arise.

Problem I,2: *From a given point to draw a straight line equal to a given straight line.*

Discussion. If a point *A* and a straight line with end points *B* and *C* are given, then it seems obvious to proceed as follows: pick up the line *BC* with a compass and draw a circle around *A* with radius *BC*. Draw any line through *A* that intersects that circle (at a point *P*, say). Then, the line *AP* seems to solve the problem. - However, this is not a solution to the problem, since the *mechanical tool of a compass* was used in the proof and such tools are not available in Euclidean geometry! Notice also that 'drawing lines' in Euclidean geometry does not mean to draw lines using mechanical tools, e.g., with a ruler, but refers to a mental act. This means that we have to work with *concepts* instead.



EUCLID*'s solution to problem* **I**,2: Let *A* be the given point and let *B* and *C* be the endpoints of the given straight line. As was shown in problem **I**,1, one can describe an equilateral triangle on the straight line *AB*. Let *D* be the third vertex of this triangle. According to postulate 3, there is a circle with radius *BC* around point *B*. According to postulate 2, the line *BD* can be extended so that it intersects the circle at a point *G*, and such that *B* lies between *D* and *G*. Clearly, *BC* and *BG* have equal length. According to postulate 3, there is also a circle with radius *DG* around *D*. According to postulate 2, the line *AD* may also be extended so that its extension intersects this second circle at a point *L*, and such that *A* lies between *D* and *L*. Since the lines *DG* and *DL* are of equal length, as are the

lines *DB* and *DA*, their differences *BG* and *AL* are also equal according to the 3rd common notion. Thus, *BC* and *AL* are of equal length, QOF.

The solution to problem **I**,2 shows that the *free moveability* of finite straight lines is guaranteed in Euclidean geometry. This is the actual purpose of this problem. So, in **I**,2, the following generalization of postulate 3 is proved:

> *For any three points A, B and C, a circle with the centre A may be described such that its radius is congruent with the straight line BC (i.e., is equal in length to the straight line BC).*

But notice that, in the proof of **I**,2, one needs the following, slightly more precise version of postulate 2 (see also footnote 2):

> *For any three pairwise different points A, B and C, the straight line AB may be extended to a longer straight line that intersects the circle around B with radius BC at two different points*.[5]

Finally, let us look at the proof of the first congruence theorem for triangles.

Theorem I,4: *Triangles are equal if two sides and the angles enclosed between these sides are equal,*

Proof: Let two triangles with the vertices *A,B,C* and *D,E,F* be given and assume that the straight lines *AB* and *DE* are congruent, and that the straight lines *BC* and *EF* are as well. Assume that the angles between these lines are equal. We have to show that the straight lines *AC* and *DF* are also congruent.

From **I**,2, it follows (due to the free moveability of finite straight lines) that the line *DE* can be placed onto *AB* and the line *EF* onto *BC.* The included angle between *DE* and *EF* was also placed on the angle between the lines *AB* and *BC,* since these angles are 'equal' according to our assumption. Due to the 9th common notion, the straight line connecting *A* with *C* is unique. Therefore, the line connecting *D* with *F,* when the specified displacement is carried out, must fall onto the line *AC*. Hence, both triangles are congruent, Q.E.D.

EUCLID 's proof is not entirely conclusive, since it remains unclear how the equality (i.e., the congruence) of angles can be shown. EUCLID uses the word *epharmózein* (ἐφαρμόζειν, 'to superpose', 'to put one upon the other'), which is undefined in Euclidean geometry. [The word is formed from the preposition ἐφ (on top of) and the verb άρμόζειν ('to join', 'to connect')]. Here, the word cannot mean a physical movement, since the points and lines to be moved are 'without parts', resp. 'without breadth', and therefore do not belong to the world perceivable through the senses. On the other hand, one cannot really imagine points

[5] The translations of the second and third postulates proposed by CLEMENS THAER in his edition of EUCLID*'s 'Elements'* (Wissenschaftliche Buchgesellschaft, Darmstadt 1962) are somewhat imprecise. In the 2nd postulate, EUCLID is not asserting that one can 'somehow' extend given distances a little, but rather that one can extend them *beyond any measure* (i.e., 'indefinitely'). In the 3rd postulate, it should not be asserted that one can draw a circle around every point *A* with every line *BC* as radius, but rather only around every point *A* with an arbitrary radius of the form *AB*. The translation of *diastêma* (διάστημα) as '*distance*' is not precise.

and lines that 'have no parts', or are 'without breadth', and therefore cannot be moved 'in thought'.

JACQUES PELETIER (1517-1582) had already noticed the unsuccessful proof of **I**,4 in his edition of the Euclidean *'Elements'* (from 1557) (cf. HEATH, op. cit., Vol. 1, p. 249). PELETIER asked the question as to why, in the proof of **I**,4, the superposition of figures is allowed, but not in other proofs (e.g., of **I**,2). If it were always allowed, then many proofs could be shortened considerably. PELETIER also argued that the equality of angles could only be established if the concept of the equality of angles was defined beforehand.

ARTHUR SCHOPENHAUER (1788-1860) understood 'superposition' as something *'entirely empirical'*, and wrote, in the 2nd volume of his work *'Die Welt als Wille und Vorstellung'* (Sämtliche Werke, Edition P. DEUSSEN, 1911, Volume 2, p. 143), that *"only physical bodies"* are movable, and therefore the process of 'superposition' does not belong to pure geometry. (It was only in structuralism that it was possible to introduce movements as automorphisms of the entire structure, i.e., as purely mathematical objects - cf. Chapter 19).

In his *'Principles of Mathematics'* (Cambridge 1903, pp. 404-405), BERTRAND RUSSELL (1872-1970) commented on these inconsistencies in EUCLID*'s 'Elements'* and wrote, somewhat mockingly, that the *'Elements'* were probably wrongly praised for their alleged logical rigour. Concerning the proof of proposition **I**,4, he wrote

> *„indeed Euclid's proof is so bad that he would have done better, to assume this proposition as an axiom,"*

- just as DAVID HILBERT had done earlier in his *'Grundlagen der Geometrie'* (1899, 10th edition 1968, op. cit., p. 14, Axiom III, 5)

RUSSELL also complained that, in this proof, EUCLID treated triangles as rigid bodies and had based it all on the concept of motion. HILBERT succeeded, in 1899, in his axiomatization of 'elementary' geometry, in completely eliminating the reference to sense-perception in the proofs and carrying out all argumentations purely conceptually (cf. Chapter 20). Using the congruence axioms, as formulated by HILBERT, the proof of **I**,4 could be repaired.

4.9 Discussion

Our brief look at the *'Elements'* of EUCLID shows us that some problems and theorems have been treated with admirable ingeniousness and complete accuracy (e.g., **I**,2), and that, in others, the argumentation is not always entirely conclusive (e.g., **I**,1 & I,4)**.** In the treatment of problem **I**,1, only the concept of *continuity* was missing, a concept that was not formulated in its full meaning and with the necessary clarity until the 19th century. EUCLID did not yet have this concept at his disposal, and therefore he cannot be reproached for overlooking the problem of continuity. The weak point in the proof of Theorem **I**,4 concerns the handling of angles. Here, it is evident that Euclidean geometry, despite intensive efforts, has not yet been freed from all *'Erdenrest'*, because it has to rely on the movability of rigid bodies. Not all ideals of an exact representation of geometry have yet

been achieved. But, all in all, EUCLID*'s 'Elements'* is an admirable work that can still inspire enthusiasm today.

PROCLUS (ca. 411-485), in his commentary on the first book of *'Elements*', saw an influence of Platonic philosophy everywhere. This commentary contains many valuable remarks and historical testimonials, but obscures the purely mathematical achievements of EUCLID. PROCLUS wrote his commentary at a time when emerging Christianity was threatening to extinguish the old ancient (pagan) culture, and, to a large extent, had done so. In a counter-movement (which started in the middle of the third century of our era), there was an attempt to revive the old Platonic doctrine, which led to the so-called *'New Platonic doctrine'.* The work of PROCLUS belongs to this New Platonic movement, which tried to place the *'Elements'* of EUCLID within the tradition of Platonic thought. However, PROCLUS says very little about the inner-mathematical objectives that led to the present form of EUCLID*'s 'Elements'.*

The axiomatic structure of EUCLID*'s 'Elements'* was used to great effect *(cf.* HERMANN SCHÜLING, 1969, op. cit.). It is most likely that, even before EUCLID, there were individual treatises of mathematical disciplines on an axiomatic basis. It is undisputed that mathematicians and philosophers strongly influenced each other in the development of the axiomatic method. Thus, in EUCLID*'s* axiomatics, there are also very clear traces of the Aristotelian way of thinking.

The Aristotelian theory of science, which we discussed in Chapter 3, is also a form of axiomatic theory. But it is substantially different from the Euclidean axiomatic theory. The objects of Aristotelian geometry are obtained through *abstraction*; in contrast, the objects of Euclidean geometry are *idealizations*.

The hypotheses on which an Aristotelian-conceived geometry can be based must (and can) be 'true', since they refer to objects of the sense-perceivable world. However, fully general universally valid theorems cannot be proven here, since they always depend on the structure of the real natural world.

In contrast, the postulates (or axioms) underlying Euclidean geometry are neither 'true' nor 'false', since the idealized objects do not belong to the real world. What is true in this world of idealized objects is determined by the postulates. On the basis of the postulates, geometry can be developed independently of epistemological problems.

The axiomatic method, which was first used in EUCLID*'s 'Elements',* has been thoroughly revised in modern times, and finally found its ultimate form in the decades around 1900. We will report on this in Chapters 10, 18, 19 and 20.

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Chapter 5
Finitism in Greek Mathematics

In today's mathematics, the infinite is present almost everywhere, and it could hardly be expressed better than the way HERRMANN WEYL described it:

> *"Will man ... ein Schlagwort, welches den lebendigen Mittelpunkt der Mathematik trifft, so darf man wohl sagen: sie ist die Wissenschaft vom Unendlichen."*
>
> [If one wants a keyword that hits the living center of mathematics, one may well say: it is the science of infinity.]
>
> HERMANN WEYL, op. cit., 1927, p. 54.

However, mathematics seems to have only gradually found this *'living centre'* in modern times. In earlier times, the concept of infinity was considered a difficult and problematic concept. Actual infinite quantities did not play a major role from antiquity until early modern times. Whenever one spoke of infinity in mathematics, it was almost always only in the syncategorematic[1] sense as 'potentially infinite'. CARL FRIEDRICH GAUSS (1777-1855) expressed himself in this sense:

> *" ... so protestiere ich zuvörderst gegen den Gebrauch einer unendlichen Größe als einer Vollendeten, welcher in der Mathematik niemals erlaubt ist. Das Unendliche ist nur eine »façon de parler«, indem man eigentlich von Grenzen spricht, denen gewisse Verhältnisse so nahe kommen als man will, während andern ohne Einschränkung zu wachsen verstattet ist."*
>
> [... so I protest first of all against the use of an infinite quantity as a terminated one, which is never allowed in mathematics. The infinite is only a 'façon de parler', in that one actually speaks of limits to which certain relations come as close as one wants, while others are allowed to grow without restriction.]
>
> C.F. GAUSS, in a Letter to SCHUMACHER, in Volume 8, pp. 216 ff, of the '*Gesammelte Abhandlungen*' by *Gauss*, Berlin 1932.

In this letter, GAUSS even wrote that it would be presumptuous to regard the *'infinite as something given.'*

[1] *Categorematic* is a word that is meaningful in itself and, hence, also has a signification (a meaning, a sense) outside of any linguistic connection. *Syncategorematic* is an expression that is not meaningful in itself and has a meaning only in connection with other words. Κατηγόρημα is the subject of the speech, i.e., what is addressed. Κατηγορέω means *'I indicate'*.

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Science Networks. Historical Studies 62, https://doi.org/10.1007/978-3-031-27304-9_5

In reviewing the question of whether infinity in mathematics can be more than a *'façon de parler'*, we find that mathematicians apparently had very different opinions at different times. It is therefore an interesting task to examine whether the use of the infinite in mathematics is entirely problem-free, or whether it must perhaps be subject to some restrictions.

In this chapter, we want to examine how the mathematicians of antiquity dealt with the infinite. They certainly did not avoid it completely, because, in the theory of *parallel lines*, in the proof of the *irrationality* of $\sqrt{2}$, in the applications of the *method of exhaustion*, etc., the infinite is touched upon. It is also touched upon in the well-known prime number theorem: *'There are infinitely many prime numbers'*. But here, the infinite appears only as potential infinity, and not as an actual infinity. This distinction goes back to ARISTOTLE. Let us first make clear what is meant by this.

5.1 Actual and potential infinity (ARISTOTLE)

ARISTOTLE discusses the *concept of infinity* in the 3rd book of his *'Lecture on Physics'* (Φυσικὴ ἀκρόασις). First of all, he states that all objects of the natural real world are limited in size and that, therefore (due to his views on the nature of mathematical objects - cf. Chapter 3), there are no infinitely large objects in mathematics. But ARISTOTLE acknowledges that, in a certain sense, the infinite does exist, namely, in the flux of time, which, in his opinion, is in continuous succession of change without beginning and without ending, etc., and also in the series of natural numbers, etc. (Book III, Ch. 5, 206a9-12).

According to ARISTOTLE, there are no infinitely long straight lines in the real world. He argues (in the third book of his *'Lecture on Physics'*, Chapter 7) that it therefore would not be a restriction on geometry if only straight lines of finite length, surfaces of limited size, etc., were allowed as objects. This is largely correct, because, in Euclidean geometry (with very few exceptions, e.g., problem **I**,12), the only straight lines that are considered are those that appear as lines connecting two different points, and hence are finite.

ARISTOTLE is faced with the dilemma that he is convinced of the non-existence of unlimited, infinite objects in the natural real world, and also in mathematics, but, at the same time, he must acknowledge that infinity somehow nevertheless does exist, because, otherwise, e.g., time would have a beginning and an ending, in arithmetic, the series of numbers would have an end, and in geometry, there would be indivisible segments of straight lines, etc. (cf. Chapter 6 of the third book of the *'Lecture on Physics'*). ARISTOTLE writes that it appears

> *"as if the existence of the infinite can neither be affirmed nor denied."*

ARISTOTLE discusses the two sides of the question as to whether the infinite exists or not in great detail. In doing so, he comes to an apparent hopelessness. But then, he finally succeeds in a remarkable way. He presents the following thesis:

> ***Thesis*:** *The infinite does not exist as something actually existing, but it may exist in the mode of possibility.*

He distinguishes between the *'actual infinite'* (*infinitum in actu*, the *categorematic* infinite) and the *'potential infinite'* (*infinitum in potentia,* the *syncategorematic* infinite).

He does not want the word 'potential' (δυνάμει, i.e., according to possibility) to be understood here in the colloquial sense, as when one says, for example, that a marble block contains a statue as its 'potential' (cf. ARISTOTLE: *'Lecture on Physics',* III. book, § 6, 206a18-21). One could produce the statue in finitely many steps; but of the infinite series of numbers, one can produce only finitely many numbers in a finite amount of time, i.e., writing them down in words or signs (cf. also JAAKKO HINTIKKA (1966, op. cit.), J. LEAR (1979/1980, op. cit.) and L. REICHE (1911, op. cit.).

According to ARISTOTLE, one cannot speak of infinity in the *categorematic* sense, but only in the *syncategorematic* sense, because the word 'infinity' cannot refer to any single existing object or any completed collection of objects. Also according to ARISTOTLE*,* one cannot speak of infinity in the *collective* sense, because, in his opinion, there are no infinite completed multitudes (or collections) of individual things. One can only speak of infinity in the *distributive* sense, i.e., of individual things in finite, but arbitrarily large, numbers.

In Greek mathematics, infinity is addressed in the *prime number theorem*, in the proof of *irrationality* of $\sqrt{2}$ and in the concept of *parallelism*. But infinity is also touched upon in all problems solved through the *method of exhaustion*, i.e., in squaring the circle (DEINOSTRATOS, ARCHIMEDES and others), in squaring the parabola (ARCHIMEDES), in determining the volume of a pyramid and a cone (DEMOCRITUS, EUDOXOS),[2] etc. The question arises as to the form in which the infinite appears, as an actual infinity or merely as a potential infinity.

In the following, we will discuss all of these problems in all necessary details. First, in Sections 5.2 and 5.3, we deal with questions about the use of *infinite quantities*, then, in 5.4 and 5.5, we deal with questions about the use of *infinite multitudes*, and finally, in 5.6 and 5.7, we deal with questions about the use of *infinite processes*.

5.2 Drawing perpendicular straight lines in the *'Elements'* of EUCLID

In Euclidean geometry, as a rule, only straight lines of finite length occur. In the postulates in the first book of the *'Elements',* for example, it is not required that one be able to draw infinitely long lines. It is required only that one *'can draw the line from any point to any other point'* (postulate 1). Definition 3 says that lines have finite length because they have extremities, and these extremities are points. But in postulate 2, it is asserted that *'every straight line of limited length can be continuously extended (as far as one likes) in a straight line'* (cf. also Chapter 4, sections 4.1 & 4.8). This means that all straight lines in Euclidean geometry are *potentially infinite.*

It is surprising that EUCLID nevertheless occasionally uses *actual-infinite,* unlimited straight lines. He does so, for example, in problem **I**,12, in which, to a given infinite straight line from a given point that is not on it, a perpendicular straight line is to be drawn.

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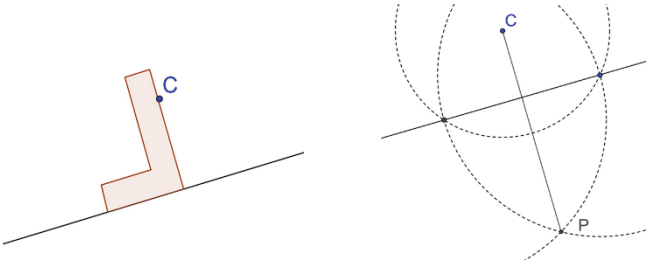
[2] DEMOCRITUS (ca. 460-370 BC) had already formulated the rule 'base area times height divided by 3' for the calculation of the volume of a pyramid. However, the correctness of the formula was proved using infinitary methods that did not exist before EUDOXOS, who lived from about 400 to 347 B.C.E., - see EUCLID: *'Elements',* Book 12, § 10. See also DEHN*'s* solution of HILBERT's third problem: MAX DEHN: *'Über raumgleiche Polyeder',* Nachr. Göttingen 1900, pp. 345-354, and MAX DEHN: '*Raumteilungen'*, Math. Ann. 55 (1900), pp. 465-478.

According to tradition, it was OINOPIDÊS (Οἰνοπίδης, circa 440 B.C.E.) who discovered how to use a compass and a ruler (i.e., an unmarked straightedge) to draw a perpendicular straight line (i.e., a so-called *'plumb line'*) from a given point that is not on a given straight line. Until then, it was common practice to insert a gnomon, i.e., right angle, so that one leg touches the given point and the other leg stands in an upright position on the given line. However, such a construction takes place in the realm of the sense-perceptible, and therefore cannot have its place in the *'Elements'* of EUCLID. In problem **I**,12, EUCLID preferred to use the construction of OINOPIDES, which is manageable with the help of concepts. It is the construction still used today.



Problem **I**,12 cannot be solved if the given point *C* is not 'above' the given finite straight line. This could easily be guaranteed with the help of postulate 2 (in the formulation as given in Chapter 4, section 4.8). But, then, the formulation of the problem would be a bit more complicated. Therefore, EUCLID assumed, without further ado, that the perpendicular 'plumb line' is to be dropped on an actual infinite, *'unlimited straight line'*: ἐπὶ τὴν δοθεῖσαν εὐθεῖαν ἄπειρον (...). Whether it is legitimate to allow for infinitely long, unlimited lines in geometry, however, is not discussed by EUCLID.

In his commentary on the first book of the Euclidean *'Elements'*, PROCLUS deals extensively with the use of unlimited, infinite lines in geometry (op. cit., p. 363-365). He admits that there are no infinite quantities in the natural real world and refers to the *'immortal Aristotle'*. But he thinks that *'the infinite exists in our imagination'* and writes (op. cit., p. 364),

> *"The infinite is thus not the object of the cognitive imagination, but rather of the unlimitedly wandering non-cognitive imagination, which addresses that as infinite, from which it must cease, since it is immeasurable, and which it cannot conceptually grasp by thinking."*

According to PROCLUS, the infinite is, hence, a fiction that exists only in the imagination of the people. He continues as follows (op. cit., p. 365):

> *"Therefore, if we place the line given as infinite, as well as all other geometric forms, such as triangles, circles, angles, lines, into the imagination (conception), will we not wonder how there is a line that is in reality infinite and that, despite its infinite extension, fits into the limited objects of knowledge? But the mind, from which the concepts and the proofs originate, does not use the infinite for proof: For the Infinite cannot be grasped by science at all, but is merely a hypothesis."*

In geometry, according to PROCLUS, one may deal with imagined objects and use the infinite as a *fiction*, as long as one can describe a general fact, which, in each concrete individual case, only refers to the limited. Indeed, in **I**,12, the actual-infinite line is only used for the (elegant) formulation of the problem; in its solution, with the help of postulate 2, only limited quantities are needed.

PLOTINUS (205-270 o.c.) holds a similar opinion. For him, it is by no means contradictory to speak of infinite lines, because

> ἢ ἐν τῷ λόγῳ τῆς αὐτογραμμῆς οὐκ ἔνι προσνοούμενον πέρας.
> [*in the notion of a line there is per se no thought of boundary.*]
> PLOTINUS: Περὶ ἀριθμῶν (*'Of numbers'*), op. cit., pp. 210-211.

But that which exists in our mind as thought or imagination cannot always be bound to existing things, as PLOTINUS acknowledges.

The gates to infinity have thus opened by a small gap. In contrast to ARISTOTLE, who only allowed for the infinite as something finite that can be enlarged at will, PROCLUS and PLOTINUS considered allowing for the infinite at least as a mental fiction.

Infinite straight lines were first used systematically by GIRARD DESARGUES (1591-1661) in his *Projective Geometry* (1638). In *Analytical Geometry* (1637), drafted by RENÉ DESCARTES (1596-1650), the straight lines are described by linear equations, and because it would be unnatural to limit algebraic descriptions to straight lines of finite length, it has finally emerged, in the course of time, that, in geometry, straight lines should always be understood as unrestricted, infinite straight lines. For DAVID HILBERT (1862-1943), in his *'Grundlagen der Geometrie'* (1899), all straight lines are infinite. With these stronger ontological assumptions, geometry can be constructed much more elegantly and simply than with EUCLID.

5.3 The concept of parallelism (EUCLID)

In Euclidean geometry, as a rule, only straight lines of finite length are mentioned. Only in the formulation of problem **I**,12 (but not in its solution) are unlimited lines used. It would be quite natural to use infinite lines in the theory of parallel lines as well, but, here, EUCLID prefers to allow only lines of finite length. Using unlimited lines, it would be easy to formulate the *postulate of parallels* (postulate 5 in the first book of *'Elements'*) directly as follows:

> (#) *Two infinite straight lines, which are intersected by a third straight line in such a way that on one side the two internal angles together are less than two right angles, intersect on the side where the angles together are less than two rights*.

But EUCLID formulated the *postulate of parallels* differently to avoid the use of infinite straight lines. First, he defines, in the 1st book of his *'Elements'* (op. cit., Def. 23):

> *Definition*. Two (finite) straight lines *s* and *t* lying in the same plane are called *parallel* if for all (finite) straight lines *s*', which extend *s,* and for all (finite) straight lines *t'*, which extend *t,* it holds that they have no point in common (i.e. do not meet).

We have slightly changed the wording to make clear that there are two universal quantifiers

hidden in the definition. EUCLID writes of the given finite straight lines *s* and *t* as *'being extended indefinitely'*, and not of unlimited lines. He uses the adverbial expression εἰς ἄπειρον (indefinitely). Thus, one only knows that *s* and *t* are parallel if one also knows that all extensions (and there are infinitely many) of *s* and *t* do not meet. Although it is quantified over infinitely many straight lines, the straight lines themselves are all finite. This shows that the above definition only refers to the potentially infinite.

EUCLID proves, without coming into contact with infinity (Book 1, §27), that *'if a straight line falling on two other straight lines make the alternate angles equal to one another, the two straight lines will be parallel to one another'*.

Since the converse of this statement could not be proved, EUCLID postulates its validity. He formulates it (somewhat artificially) as follows:

> POSTULATE OF PARALLELS: *For every two finite straight lines that are intersected by a third straight line in such a way that on one side the two inner angles together are smaller than two right angles, there are straight extensions that intersect, the intersection point P being on the side where the two angles are located, the sum of which is smaller than two rights.*

The parallel postulate thus serves to mark the parallelism. In its formulation, only the potentially infinite is used (and quite consciously).

It is remarkable, however, that the elimination of the actual infinite, which was still used in our previous formulation (#), has come at a price. While, in (#), the point *P* is definable as the intersection of the two given lines, and thus can be used in all further constructions, the intersection *P* has not been constructed in the Euclidean formulation of the parallel postulate. It is only stated that the intersection *P* must exist *'per se'*.[3] It is noteworthy that, this early in Greek mathematics, not every proof of existence was led by a construction of the object in question (more on this in Chapter 17).

The discrepancy between statements **I**,12 and postulate 5, in which infinite straight lines are used, resp. are avoided, suggests that one of the two formulations was not written by EUCLID and was only interpolated by a later editor of the *'Elements'*. This assumption cannot be proven. But even if the formulation of **I**,12 or of postulate 5 did not come from EUCLID, both formulations are nevertheless from antiquity.

5.4 The number of grains of sand (*'The Sand-Reckoner'* of ARCHIMEDES)

We now discuss, in 5.4 and 5.5, questions about the use of *infinite multitudes* in Greek mathematics, starting with the question as to the sense in which the multitude of all natural numbers is infinite.

It was very difficult for the mathematicians of antiquity to approach infinity at all. With the ordinary numerals of the Alexandrian numerical system, if one does not want to force the Greek language, one can only count up to 100 000 000 (=10^8, μύριαι μυριάδες). In 'normal, daily life', it was not necessary to be able to name even larger numbers.

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[3] PROCLUS writes, in his *'Euclid Commentary'* (op. cit., p. 294), that the parallel postulate represents only a property of right angles, but the postulated existence of the intersection is not used in any construction.

Instead of the Indian-Arabic numerical characters (ciphers) that are common today, the Alexandrian numerical system was based on Greek letters:

1 = α, 2 = β, 3 = γ, 4 = δ, 5 = ε, 6 = ϝ (Digamma), 7 = ζ, 8 = η, 9 = θ,

With the character ι for 10, the numbers 11, 12, 13, ..., 19 are represented as summations by writing the summands next to each other: 11 = ια, 12 = ιβ, 13 = ιγ, ..., 19 = ιθ. With the character ϰ for 20, the numbers 21 = ϰα, 22 = ϰβ, 23 = ϰγ, ..., 29 = ϰθ are formed. Furthermore, λ stands for 30, μ for 40, etc., and ϱ for 100, σ for 200, etc. With further auxiliary characters, one can continue counting and name all numbers smaller than 10^8 (see GEORGES IFRAH, *'Universalgeschichte der Zahlen'*, Frankfurt/M 1989, p. 289 ff). Larger numbers were 'unnamable'.

But the poets liked to allude to 'unnamable' or 'unattainable' numbers, such as the number of grains of sand on the earth, the number of stars in the night sky, the number of sea waves, the number of locusts, etc. Some poets even had fun with the naming of such oversized numbers, such as ARISTOPHANES, who (in the *'Acharnians'*) even wrote of the product of desert sand times sea sand. However, the 'sand number', i.e., the number of grains of sand on earth, became a prime example of an extremely large and, in some ways, 'unattainable', and therefore 'infinitely large', number.

The 'number of sand' can be found as early as HOMER*'s 'Iliad',* in the 9th chant, verse 385, and, even in later times, poets and philosophers (PINDAR, PLATO, KALLIMACHOS, OVIDIUS, CATULLUS, SENECA et al.) repeatedly used the 'number of grains of sand' as a symbol of a number that is so large that it seems to reach infinity, one that it is unknown to humankind and will always remain so.[4]

But the Pythian Apollo, the god of prophecy, announced in an oracle that he knew the number of grains of sand in the whole cosmos. HERODOTUS (ca. 480-428? B.C.E.) handed down, in the 1st book 'Klio', § 47,3, of his work *'Exposition of the history'* (ἱστορίης ἀπόδειξις), what Apollo was communicating. It begins with the following hexameter:

Οἶδα δ᾽ ἐγὼ ψάμμου τ᾽ ἀριθμὸν καὶ μέτρα θαλάσσης, ...

[Verily I know the number of the grain of sand and the measures of the sea, ...]

The oracle of Delphi had announced this verse to CROESUS, king of Lydia, when he wanted to know if he could start a war against the Persian king CYRUS*, the Elder*. He started this war in the year 558 B.C.E., and, from this date, one can deduce that the response of the oracle had been given shortly before (thus about 560 B.C.E.).

About 320 years later, around 240 B.C.E., the mathematician ARCHIMEDES (287-212 B.C.E.) was asked by GELON, the eldest son of the Sicilian King HIERON II, whether the number of grains of sand is infinite, *'as some believe',* or, on the contrary, whether it is finite. GELON probably knew the statement that the oracle in Delphi had proclaimed, and only wanted to know whether humans could also know the number of grains of sand and

[4] PINDAR in the 2nd and in the 13th *'Olympic Hymn'*, PLATO in the *'Euthydemos',* 294b, SENECA in his tragedy *'Medea',* III,1, line 557*,* KALLIMACHOS in the hymns H3, line 253, as well as H4, lines 28 and 175, CATULLUS in the poems 7 and 61 etc., OVID, *Tristia* I,5 and the *'Art of Love'* and later again with CHRISTOPH MARTIN WIELAND in his *'Musarion'*, etc. etc.

whether ARCHIMEDES could calculate this number.

ARCHIMEDES felt challenged, and wrote a long treatise titled *'Psammitês'* (Ψαμμίτης, Lat.: *'Arenarius',* Engl. *'The Sand-Reckoner'*), in which he developed a Notation System for numbers that goes so far that the number of grains of sand in the whole cosmos (if it could be known) can be expressed within this system, and is therefore finite (see ARCHIMEDES: *'Opera Omnia'*, Volume II, pp. 242-291). ARCHIMEDES proceeded as follows.

He started out from the usual Alexandrian Notation System and called all numbers that can be expressed there, which, in today's notation, are all numbers that are smaller than 10^8, *'numbers of the first order'.* The number 1 is called the *'unit of the first order'.* The first number (that is, the number 10^8), which is no longer expressible in the Alexandrian system, he called the *'unit of the second order'.* With this and the notations for the numbers of the first order, he could express numbers that he called *'numbers of the second-order'.* These are the numbers between 10^8 and 10^{16}, where 10^{16}, however, is the *'unit of the third order'.* 10^{24} is the *'unit of the fourth order',* etc. In this way, by introducing more and more new units of higher orders, it is finally possible to denote all numbers that are (in today's notation) smaller than $(10^8)^{(10^8)}$, cf. G.H.F. NESSELMANN, op. cit., pp. 122-125. All numbers that can be denoted in the way described so far are called *'numbers of the first period'.*

By using a name for the first number that follows all numbers of the first period, the numbers of the *'second period'* can be denoted, etc. By introducing ever new names for the boundaries of the individual periods and by combining all the periods into a *series,* etc., one can create number-systems that go further and further. But the range of each individual system is limited.

Using some astronomical data, ARCHIMEDES finally found that there is room for no more than 10^{51} grains of sand on the globe and 10^{64} grains of sand in the entire cosmos. The number of grains of sand on the earth is therefore already expressible with a number of the 1^{st} period (and the 7^{th} order), and is therefore finite.

The Greeks did not establish a System of Notations for numbers based on a fixed finite alphabet of characters that allows for the representation of all numbers. However, the Babylonians had already succeeded in doing so in the Seleucid period, as did the Indians around the 6^{th} century. They designed a *'positional notation system',* also called a *'place value system'*, which works with a finite supply of signs and allows for the formation of notations for all finite numbers. The main idea here was the use of a sign to designate unoccupied places. Today, we are well acquainted with the Indian decimal system, which works with the ten digits 0, 1, 2, ..., 9 and which became known in Europe through the mediation of AL-KHWOARIZMI (ca. 780-850).

We return to the discussion of the concept of the number in Greece. Here, again, the discrepancy between the Platonic and Aristotelian views becomes apparent. In the Aristotelian view, numbers are only linguistic constructs, and therefore - as the investigations of ARCHIMEDES show - only form a *potentially infinite* multitude (cf. Chapter 3). In the Platonic view, numbers are sets of indistinguishable units and are not the same as numerals. The numbers are related to ideas and belong to the realm of Being. In this

realm of eternal Being, all numbers are present, uncreated and imperishable. They form there, in the realm of being, an *actual-infinite* totality (cf. Chapter 2).

EUCLID largely followed the Platonic view in number theory and, in the 7th book of his *'Elements'*, also defined numbers as (finite) 'sets (or multitudes) of units'. He did not set up a System of Notations for numbers. In this respect, like PLATO, he does not assume that we have to construct the numbers one after the other, but that they are all somehow predetermined. One may conclude that the series of natural numbers, for EUCLID, is an *actually infinite multitude.* - But what about the series of all prime numbers?

5.5 The existence of infinitely many prime numbers (EUCLID)

One can communicate the classical prime number theorem of EUCLID in the form

(†) *There are infinitely many prime numbers.*

Then, it is asserted that the prime numbers all exist somewhere and somehow, and that they form altogether an actual-infinite totality. - But EUCLID himself formulates the theorem quite differently, namely,

> Οἱ πρῶτοι ἀριθμοὶ πλείους εἰσὶ παντὸς τοῦ προτεθέντος πλήθους πρώτων ἀριθμῶν.
> [*There are more prime numbers than any displayed multitude of prime numbers.*]
> EUCLID: *'Elements',* Book 9, § 20.

The 'displayed multitudes' here are finite sets, as the colloquial use of the word τὸ πλῆθος (multitude) indicates. In this formulation of the proposition, it looks as if the infinite is treated only as a potentially infinite multitude.

EUCLID proves the prime number theorem as follows: for every finite set of prime numbers $p_1, p_2, ..., p_n$, every prime divisor q of $z = 1+(\Pi p_i)$ is obviously different from all primes $p_1, p_2, ..., p_n$, Q.E.D.

By the way, one can see that EUCLID formulated exactly that as a proposition that he proved: For every displayed (finite) multitude of prime numbers, there is at least one more prime number. So, the word 'infinite' is not artificially circumvented here!

A short digression: In 1873, GEORG CANTOR proved, in set theory, that the set $\mathbb{R}$ of all real numbers is uncountable. This formulation does not exactly reflect what CANTOR proved. He only proved (with a diagonal argument, credited to PAUL DU BOIS-REYMOND) that there is at least one more real number for each displayed countable sequence of real numbers. In EUCLID*'s* words, we could also say: *there are more real numbers than any given countable set of real numbers.* In this formulation, CANTOR's theorem only states that the totality of all real numbers is potentially uncountable. Whether this totality is an independently existing (finished) thing is a different matter, and cannot be deduced from CANTOR*'s* proof (see also Chapter 15).

We return to the prime number theorem. - In EUCLID's theorem, one can, if one wishes, very easily avoid the reference to 'finite multitudes', and assert simply that, for every prime *p*, there is a greater prime *q*. For *q*, one only has to choose a prime divisor of 1 + *p*! (Here, *n*! is, in the notation of CHRISTIAN KRAMP, Cologne 1808, the product of all numbers *z*

with $z \leq n$.) The following, somewhat stronger theorem is also of interest here, since the bound for the 'next' prime after p is much lower than $1 + p$! (this was first proved by ADRIEN-MARIE LEGENDRE in 1798):

> **Theorem**: *If* p *is a prime number, then any prime divisor of* 2^p-1 *is strictly larger than* p. *Hence, for every prime number there is a greater prime number.* [5]

These considerations show that, although EUCLID, in his prime number theory, only makes use of the potentially infinite, he does not consciously avoid the actual infinite.

5.6 The exhaustion method (EUDOXOS)

In the twelfth book of the *'Elements',* EUCLID gives some applications of the method of exhaustion, as it was first conceived by EUDOXOS (ca. 400-347 B.C.E.) and later worked out with more accuracy by ARCHIMEDES. The method is based on the so-called *'axiom of* EUDOXOS', which is also called (somewhat historically inaccurately) the *'Archimedean axiom'* (cf. *Euclid*: *'Elements',* Book V, Def. 4, and ARCHIMEDES' introduction to his treatise on the *'squaring of the parabola'* and Lambanómenon 5 in *'Sphere and cylinder'*):

> *If* A *and* B *are two magnitudes of the same kind (e.g. finite straight lines, or surfaces, or numbers, etc.), and* A *is smaller than* B*, then one can find such a multiple of* A (*written as* nA) *that is larger than* B.

This is a direct consequence of (cf. EUCLID *'Elements'*, Book X, § 1):

> *If* A *and* B *are two magnitudes of the same kind and* A *is smaller than* B*, and if from* B *one subtracts a magnitude greater than its half, and from the remainder again a magnitude greater than its half, and if this process be repeated continually, there will be left after finitely many steps some magnitude which will be less than* A.

This statement is the basis of the *method of exhaustion*. On this basis, the quadratures (of a circle, of a parabola) and the calculations of the volume (of a pyramid, a cone, a sphere) can be carried out in such a way that the infinite is present only in the sense of the potentially infinite. For example, ARCHIMEDES, in his paper *'Squaring of the Parabola',* has to calculate the following geometric series:

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[5] *Proof*: Let p be a prime and let q be a prime divisor of $2^p - 1$. Then (according to FERMAT's little theorem), $2^{q-1} \equiv 1 \pmod{q}$. Let r be the smallest natural number such that q is a divisor of $2^r - 1$, hence $r \leq q-1$ and $r \leq p$. Then, $q - 1 = xr + y$ (division with remainder) with $y < r$. Thus, $1 \equiv 2^{q-1} \equiv (2^r)^x 2^y \equiv 2^y \pmod{q}$. Since r was chosen as a minimum, $y = 0$ holds. Division with remainder yields $p-r = ar + b$ (with $0 \leq b < r$). From $2^p \equiv 1 \equiv 2^r \pmod{q}$ follows $1 \equiv 2^{p-r} \equiv (2^r)^a 2^b \equiv 2^b \pmod{q}$. Since r was selected as minimum, $b = 0$ must hold. Hence, $p = (a+1)r$. Since p is a prime number, $a = 0$ or $r = 1$. But it is $r \geq 2$. So, $a = 0$, and therefore $p = r$, and $q-1 = xr = xp \geq p$ as well. Therefore, $q > p$, and everything is proven.

$$\sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n = 1+\frac{1}{4}+\frac{1}{16}+\ldots=\frac{4}{3}.$$

To do this, it is necessary only to determine the limits of the partial series, and ARCHIMEDES finds these limits using the method of exhaustion as described above. (This corresponds to the statement made by C.F. GAUSS in the statement quoted earlier).

In the axiom of EUDOXOS, or its corollary, infinity is apparently addressed only in the mode of possibility. This is also evident in EUCLID′s use of the future tense in such rare cases: λειφθήσεται ('will remain') and ὃ ἔσται ἔλασσον ('will be smaller'), instead of λείπεται ('remains') and ὅ ἐστιν ἔλασσον ('is smaller'). SUSUMU YAMAMOTO, op. cit., has pointed this out.

5.7 Proofs of irrationality (HIPPASUS)

Any reference to the actual infinite is also avoidable in the proof of the irrationality of $\sqrt{2}$ (cf. Chapter 1). In fact, for any given square, the process of *anthyphairesis* (ἀνθυφαίρεσις, reciprocal subtraction), also called *'Euclidean algorithm'*, applied to the diagonal and the side of the square, will not terminate after a finite number of steps. But, in the process of anthyphairesis, any given straight line *g* will be undercut in length after a finite number of steps. This is sufficient to prove that the algorithm cannot lead to a common measure. Thus, infinity also occurs here (in a completely natural way), but only in the mode of possibility.

In another proof of the irrationality of $\sqrt{2}$, which ARISTOTLE sketched in his *'Analytica priora',* book I, Chapter XXIII (lines 41a26-27) and EUCLID gave with a few more details in his *"Elements"* (Book X, § 115a), infinity no longer occurred at all, for, if $\sqrt{2}$ were to be a rational number, for example, $\sqrt{2} = \frac{a}{b}$ with *a* and *b* as relatively prime natural numbers, then $2b^2 = a^2$, that is, *a* would be even, and consequently (because *a* and *b* are relatively prime), *b* would be odd; but, in a square number, the prime factor 2, if it occurs, must have an even occurrence - a contradiction! (Cf. TH. HEATH, op. cit., vol II, p. 298-300, and vol. III, p. 2, cf. also C. THAER, op. cit., p. 143 & p. 441, and pp.313-314 & p. 462.)

5.8 The exclusion of the 'unlimited'

We can conclude that the Greek mathematicians almost always wrote of infinity in their works only in the mode of possibility. The actual infinite was probably used only very rarely. But they did occasionally make use of it.

HERMANN HANKEL wrote (op. cit., 1874),

> *"Solange es griechische Geometer gab, sind dieselben immer vor jenem Abgrund des Unendlichen stehengeblieben."*
>
> [As long as there have been Greek mathematicians, they have always stopped at that abyss of the infinite,]

and JOHANN LUDWIG HEIBERG wrote (op. cit., 1925, p. 4)

> *"... daß ... die Mathematiker ...* [der Antike] *... den Begriff des Unendlichen in ihren Beweisen völlig vermieden"* hätten,
>
> [... that... the mathematicians ... (from antiquity) ... (would have) avoided completely the concept of infinity in their proofs.]

But is there a deeper reason for this avoidance? ABRAHAM ROBINSO(H)N (1968, op. cit.) thought that, in the cautious formulations of EUCLID, there is

> *„a trace of the distaste for infinity."*

HELMUT HASSE and HEINRICH SCHOLZ have even referred to a *'strict finitism'* in Greek mathematics (cf. HELMUT HASSE & HEINRICH SCHOLZ: 1928, op. cit.).

Even if some of these statements are somewhat exaggerated, it can be said that the actual infinite is very rarely found in the mathematics of Greek antiquity. However, the avoidance of actual infinity was never elevated to the status of a program by mathematicians, and so it is not correct to speak of *'strict finitism'*. But one can certainly speak of *finitistic tendencies,* because there is something deliberate about the treatment of some chapters of Euclidean geometry (for example, the theory of parallel lines) on the basis of finite straight lines.

An explicit rejection of the infinite in mathematics cannot be found in the writings of the ancient mathematicians. One finds such negative statements only with certain philosophers, for example, with ARISTOTLE (we have already mentioned this) and then again with PLOTINUS. In PLOTINUS' treatise *'Of numbers'* (Περὶ ἀριθμῶν), op. cit., p. 164-165, we read

> Τί οὖν ἐπὶ τοῦ λεγομένου "ἀριθμοῦ τῆς ἀπειρίας"; ἀλλὰ πρῶτον, πῶς ἀριθμός, εἰ ἄπειρος; οὔτε γὰρ τὰ αἰσθητὰ ἄπειρα, ὥστε οὐδὲ ὁ ἐπ᾽ αὐτοῖς ἀριθμός, οὔτε ὁ ἀριθμῶν τὴν ἀπειρίαν ἀριθμεῖ.
>
> [*But what about the so-called "number of infinity"? Meanwhile, how can it be number if it is infinite? For neither are the sense-perceivable things* (altogether) *infinite, and hence also not their number, nor can the one who counts them count up to infinity*.]

PLOTINUS means that infinite numbers have no reality, because there are (in his opinion) no totalities of infinitely many things in our natural real world, and, in addition, there is no one who could 'count' such totalities. Numbers only exist for PLOTINUS if they either appear as numbers (of real things) or are the result of counting (i.e., an act). He does not consider numbers as independent quantities (καθ᾽ αὑτό). PLOTINUS concludes (op. cit., p. 164-165):

> Ἀλλὰ τὸ ἄπειρον δὴ τοῦτο πῶς ὑφέστηκεν ὂν ἄπειρον; ὃ γὰρ ὑφέστηκε καὶ ἔστιν, ἀριθμῷ κατείληπται ἤδη.
>
> [But this Infinite, how can it even have existence as an Infinite? For what has existence and is somewhere is already grasped by number.]

The term 'infinite number' is therefore a contradictory concept for PLOTINUS. There are no infinite numbers for ARISTOTLE either, because they can be neither even nor odd (*'Metaphysics',* M8, 1084a1-6). In contrast, according to PLOTINUS, it is by no means contradictory to speak of infinite lines, because, here, the adjective 'infinite' can be

understood in the sense of potential infinity (see section 5.2). According to PLOTINUS, actual infinity does not exist either in arithmetic or in geometry.

In ARISTOTLE and PLOTINUS, we find an explicit rejection of actual infinite magnitudes, and we can, if we wish, say that both philosophers represented a *'strict finitism'*. But the Greek mathematicians never formulated such a 'strict' conception. They never explicitly limited the range of mathematical objects to finite objects. We repeat, therefore, that it is problematic to say that they too advocated a (more or less strict) finitism. But it is undoubtedly true to say that their works are based on finitist tendencies.

But what speaks against using the actual infinite in mathematics? From the point of view of the Greek mathematicians and philosophers, probably only that the concept of an actual-infinite magnitude could be contradictory, as the paradoxes of the Eleatic philosopher ZENO show, and that the concept of actual infinity in the natural real world does not seem to be interpretable. No other reservations were expressed in antiquity.

The paradoxes of ZENO were intensively discussed in antiquity, and, afterwards, in the high scholasticism (ca. 1240-1300) and the late scholasticism (ca. 1300-1450) as well. (We will report on this in the following Chapter 6.) But they could only be satisfactorily resolved in the 17^{th} century, after the concept of function and the concepts of a dependent and an independent variable were introduced. A mathematically exact theory of infinite sets could only be given in the 19^{th} century using set theory (BERNARD BOLZANO, GEORG CANTOR, ERNST ZERMELO and others). That such a theory is free of contradictions, however, cannot be proved with finite methods alone (according to the theorems of KURT GÖDEL, 1931).

The inclusion of the actual infinite in mathematics became necessary and possible only in modern times. On the basis of set theory, the systems of natural numbers, real and complex numbers, etc., could be introduced as actual-infinite systems. On the basis of these systems, the terms of the differential- and integral-calculus could be defined. For example, the important (and also beautiful) theorem could be proved that every holomorphic function can be developed around every point of its domain of definition into a convergent power series, i.e., it can be written as a polynomial of infinite length (and, in this respect, as a 'known' function), etc. All of these concepts require the existence of actual infinite sets for their introduction. In this respect, HERMANN WEYL*'s* statement quoted at the beginning of this chapter is unrestrictedly valid for today's mathematics.

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Chapter 6
The Paradoxes of ZENO

"The theory of infinity has its difficulties; whether one accepts the existence of an infinite or not, ... immediately one is threatened with many unpleasant consequences."
ARISTOTLE*: 'Physics Lecture',* Book III, Ch. 4, 203b32.

The concept of infinity has been considered a difficult and problematic concept from antiquity to modern times. The view that DESCARTES occasionally expressed in his letters to MERSENNE

"nostre âme, estant finite, ne peut comprendre l'infiny."
[Our soul, being finite, is unable to understand the infinite.]
DESCARTES' letter, from November 11, 1640; Œuvres de DESCARTES, vol. III, p. 234.

was widespread. People did not dare to deal with the infinite for fear of running into contradictions. One also asked oneself the question whether it can be meaningful and useful at all to deal with the infinite. Perhaps one only struggles without achieving anything, like Sisyphus, for example, who, in the lower world, had to roll uphill a huge marble block, which, as soon as it reached the top, always rolled down again - *a senseless infinite work.*

LUCIAN *(*ca. 120-180) compared the concept of infinity with Penelope's *'infinite work'*: during the daytime, she worked at a robe, and in the night, she undid the work of the day. She never finished it and did not ultimately achieve anything.[1]

VOLTAIRE (1694-1778) wrote, in his *'Dictionnaire Philosophique'* (Oeuvres complètes de VOLTAIRE (70 volumes), Volume 41, Kehl 1785, p. 305), under the keyword *'Infini'*:

"Il semble que la notion de l'infini soit dans le fond du tonneau des Danaïdes."
[It seems that the notion of infinity is at the bottom of the Danaïdes' barrel.]

The Danaïdes were the 49 daughters of the Libyan king Danaus, and they were punished in Hades by being compelled everlastingly to pour water into a perforated barrel (cf. HORACE*,* Odes book III, 11, 25-29). With this example, VOLTAIRE not only illustrated the concept of infinity, but also brought it - just like LUCIAN - close to the useless and unavailing. In the end, VOLTAIRE gave the good advice to stop worrying about infinity and

[1] LUKIAN: *'Die entlaufenen Sklaven',* In: Sämtliche Werke, G. Müller Verlag München 1911, Volume 2, p. 419.

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U. Felgner, *Philosophy of Mathematics in Antiquity and in Modern Times*,
Science Networks. Historical Studies 62, https://doi.org/10.1007/978-3-031-27304-9_6

to think about one's own health:

> *"Il vaut mieux sans doute penser à sa santé qu'à l'espace infini."*
> [Doubtless, it will be better to think about one's own health than about the infinity of space.]

In the following, we will nevertheless deal with the infinity problem. We will, however, limit ourselves to the discussion of the paradoxes of ZENO and their history of reception and influence.

6.1 The Zenonian Paradoxes

In the fifth century B.C.E., the pre-Socratic philosophers began to deal with the problem of what the basic concepts of cosmology might be. They soon came across the problem of whether *sense-perception* or *conceptual thinking* is more persuasive.

PARMENIDÊS (Παρμενίδης, born ca. 515 B.C.E.) maintained that the phaenomena of sense were delusive, and that it is only through thinking that a person can attain knowledge of that which truly exists. Only that truly exists which remains always the same. He expressed his doctrine with the familiar quotation: *'Because the same thing is to be perceived in thought, and to exist'* ("... τὸ γὰρ αὐτὸ νοεῖν ἐστίν τε καὶ εἶναι"). In his opinion, the emergence and passing of things is only an illusion of the senses. The things themselves are characterized only by their essential properties, by that what they are *per se*. That what changes and moves can be detected through the senses, but is not accessible to thinking.

His disciple ZENO (Ζήνων, born around 490 B.C.E. in Elea) tried to prove these statements. The Zenonian proofs were handed down to us by ARISTOTLE (in his *'Physics Lecture'*, Φυσικὴ Ἀκρόασις, book VI) and SIMPLICIUS (Σιμπλίκιος, in his commentary on this Aristotelian work). ZENO wants to show that one gets into egressless entanglements if one assumes that motion would be accessible to pure thinking. Thus, 'motion' cannot be grasped by thinking and, hence, is only a delusion of the senses and non-existent for pure thinking (cf. also PLATO *'Sophistes'*, 248a7-10).

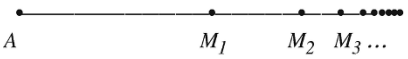
All of ZENO*'s* arguments take the form of paradoxes. However, they are faulty. Nevertheless, we want to deal with them in spite of their faults, because they have had a great effect right up to modern times. Their effect on scholastic philosophy was extraordinary.

(1) The most famous of all of ZENO*'s* paradoxes is known as the 'Paradox of *Achilles and the Turtle'* (ARISTOTLE, *'Physics Lecture'*, Z9, 239b15; DIELS-KRANZ in the section on *'Zenon'*: A26, page 253):

> *"The slowest runner will never be caught by the fastest. First of all, the chaser must reach the point from which the chased runner started, so that the slower runner always has a slight advantage."*

Achilles is the strongest, most beautiful and swiftest racer among the Greeks before Troy (Troja, Ilium). In HOMER*'s Iliad,* he's always called *'the courageous racer Achilles'*. We are told in the paradox that he of all people cannot even catch up with a turtle if it is given a small lead. If Achilles starts at point A and the turtle at point M_1, then, as soon as

Achilles reaches M_1, the turtle will have crawled a little further and reached a point M_2 at the same time, and when Achilles reaches M_2, the turtle will have reached point M_3, and so on, *ad infinitum*.



But, of course, it is not true that the slower runner *'always'* has a slight lead and that Achilles can *'never'* (οὐδέποτε) overtake the turtle, because the sequence of the ever-shorter time intervals converges towards a point of time t_0, and, as soon as this moment t_0 passes, the swifter racer is ahead of the slower one. Thus, the 'error' in the paradox is already recognized. ZENO pretends that the process of catching up can be discussed independently of the time required for this.

With this paradox, ZENO wants to prove that the process of moving cannot be understood in thought, because such a process, as an infinite process, *cannot be terminated* (at least, in thought). The proof is based on the construction of an infinite sequence of points that must be passed through one after the other.

ARISTOTLE analysed ZENO's arguments in detail in book VI of his *'Physics Lecture'*, Chapter 2 and Chapter 9 (and also in book VIII, Chapter 7). He argues against ZENO that the 'infinite' (ἄπειρον) is treated, on the one hand, with respect to extension and, on the other hand, with respect to divisibility, without distinguishing them. Clearly, no one can travel an infinitely long distance in a finite amount of time, but (in principle) everyone can travel a finite distance in a finite amount of time, even if there are infinitely many points of division marked on it.

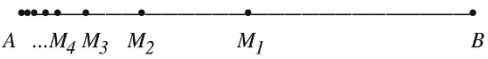
However, we will emphatically repeat that ZENO does not question the fact that the runner apparently is able to move from *A* to *B* (everyone can recognize this with the own eyes), but only claims that the phenomenon of movement is not comprehensible in thought. This is already evident from the fact that no real situation is being discussed, but only an idealization. The infinite sequence of points is a sequence of indivisible points (cf. Chapter 4) that are not sensually perceptible and exist only for thought. The story of Achill and a turtle is only a disguise that simulates the presence of a real, observable movement. In fact however, only a purely mathematical configuration is discussed.

In his proof, ZENO defines two infinite sequences of objects, a sequence of geometric points on a straight line and a sequence of points of time. Both sequences depend on each other. But, in the discussion as to whether the turtle will finally be caught by Achilles, one of the two sequences will be forgotten and only the other will remain in the argumentation and be treated as if it were independent of the first. ZENO seems to suggest that the time used in rethinking the process of catching up is always the same in each step. If this were the case, then the process of rethinking all steps of catching up would never come to an end. But this is clearly a mistake.

ARISTOTLE also suggests that the essential error in ZENO*'s* argument is that it ignores the fact that motion occurs in a *continuum* of points in space and a *continuum* of points in time. He also mentions, in his *'Physics Lecture'* V,2 (233a34), that the distance travelled *depends* on the elapsed time. However, neither ARISTOTLE nor ZENO had the *concept of a*

function, $f(t) = s$ (i.e., of *independent* and of *dependent* quantities), which would enable them to describe movements and make them accessible to thinking.

(2) The paradox of the *runner in the stadium* is somewhat more difficult to solve (ARISTOTLE, *'Physics Lecture',* book VI, Chapter 9 (Z9), 239b11-14):



A runner in a stadium starts at point *A* and wants to reach point *B*. However, he will *'never'* arrive at *B*, since he would first have to get to the middle point M_1 of the route leading from *A* to *B*, and before reaching M_1, he would have to get to the middle point M_2 of the route leading from *A* to M_1, and before reaching M_2, he would have to get to the middle point M_3 of the route leading from *A* to M_2, and so on. In order to reach point *B*, he would have to pass through the intermediate points $M_1, M_2, M_3, M_4, ...$ but in reverse order! In order to reach M_n, he must first have passed through M_{n+1}. But, since there is no first point in this reversely ordered sequence, he cannot get away from the starting point *A* at all.[2]

With this paradox, ZENO wants to prove that the process of moving cannot be understood in thought because such a process *cannot be started* (at least, in thought).

ZENO suggests that the runner in the stadium, when passing through the intermediate points ... M_4, M_3, M_2, M_1 (in that order), should reach a *first* point of that sequence, and then a *second* point, etc. This would be an admissible argument in the case of a finite number of points. But such a *first* point does not exist in the infinite point set under consideration. ZENO thus tacitly assumes that a property that is valid for finite sets of points would also be valid for infinite sets of points. This is an error in ZENO*'s* argumentation.

(3) ARISTOTLE reports upon two further paradoxes of ZENO (the *'flying arrow'* and the *'stadium'*, cf. Chapter 4, section 4.4), and SIMPLIKIOS (he lived around 530) mentions, in his commentary on the Aristotelian *'Lecture on Physics'*, the so-called *'division paradox'* (see DIELS-KRANZ, I, fragments B1, B2). Since they do not belong to the topics to be discussed here, we will not go into them.

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[2] Here, the presentation in the Aristotelian Lecture on physics (in Book VI, 239b11-14) is not clear, since it is only mentioned that the runner, who starts at point A and wants to reach point B, must first reach the point M_1 in the middle between A and B. It remains unclear how the iteration should proceed. Should the line from A to M_1 or the line from M_1 to B be halved again, and similarly in the further iterations? Both cases are plausible. However, ARISTOTLE himself refers to his own investigations conducted earlier (in book VI, Chapters 4-6), and this might indicate that he had the first case in mind. The second case would only produce a simplification of the paradox of Achilles and the turtle. We decided to understand the iteration similarly to be the process of halving in the first case, because, in this case, a paradox appears that was also intensively discussed in the Middle Ages, as we will see in the following sections 6.2, 6.3 & 6.4. (cf. also ÜBERWEG/PRAECHTER: *'Die Philosophie des Altertums',* Berlin 1926, p. 87, and also KIRK-RAVEN-SCHOFIELD, op. cit., Chapter IX).

The paradoxes of ZENO had a tremendous effect in antiquity and the Middle Ages. Very few people were able to name the mistakes in the argumentation exactly. Most of them were at a loss. Some even considered the arguments convincing. In the Middle Ages, the paradoxes again played a major role when they were paraphrased in numerous disputes. We will report on this next.

6.2 The effect of ZENO's paradoxes in the Middle Ages

Beginning in late antiquity, for a long time, neither mathematics nor philosophy had existed as independent disciplines in Europe. It was only from the 11th or 12th centuries onwards that both awakened to a new life. For a while, philosophy saw itself as *ancilla theologiae,* as a servant of theology. The aim was to integrate the 'new' Christian faith into the overall construction of the 'old' (pagan) sciences. One knew that one could not rationally justify the Christian dogmas, but one tried to explain them better by ratio. In particular, one tried to reconcile the beliefs with the great philosophical thought constructs of PLATO and ARISTOTLE, or - if necessary - to defend them against these philosophical teachings. But, from the end of the 12th century onwards, philosophy and, in particular, natural philosophy, gradually freed themselves more and more from the fetters of theology. This led to ever greater conflicts and, finally, to an Éclat that had serious consequences in 1277.

ARISTOTLE's view that time is without beginning and without ending, and that there was not a first human being and there will not be a last one, was in contradiction to Christian doctrine that God created the world and humankind before finite time and will destroy them again sometime in the future. The Paris Synod (1210) and the papal legate ROBERT DE COURÇON prohibited, in 1215, the inclusion of Aristotelian *'Metaphysics'* and all scientific and philosophical writings of ARISTOTLE in the curriculum of Paris University, which had been founded in the meantime (founding dates: 1190-1208). The decree of 1210 contained the ban:

> *"Nec libri Aristotelis de naturali philosophia nec commenta legantur Parisius publice vel secreto."* (quoted after H.DENIFLE -A.CHATELAIN: *'Chartularium Universitatis Parisiensis',* Volume 1, Paris 1899, p. 70, No. 1).

At the Paris Synod of 1210, the main work *'De divisione naturae'* by JOHANNES SCOTUS ERIUGENA was also rejected and its reading forbidden. The decree of 1215 further states:

> *"Non legantur libri Aristotelis de metafisica et de naturali philosophia"*
> (quoted from DENIFLE-CHATELAIN, loc. cit., pp. 78-79, no. 20).

Pope GREGORY IX had already lifted these prohibitions by 1237. But the doctrine of the eternity of the world was expressly condemned in Paris in 1270. On March 7, 1277, the Bishop of Paris, STEPHAN TEMPIER, condemned 219 theses of Aristotelism and Averroism. He spoke of *despicable heresies and false fantasies: "execrabiles errores ... et insanias falsas".*[3] In particular, he condemned Aristotelian cosmology and the doctrine of the

[3] See: H.DENIFLE-A.CHATELAIN: *'Chartularium Universitatis Parisiensis',* Volume 1, Paris 1899, or else ROLAND HISSETTE: *'Enquête sur les 219 articles condamnés à Paris le 7 Mars 1277'.* Louvain-Paris 1977.

supraindividual human soul (which is incapable of sinning). Some teachings of ROGER BACON, ZEGER (=Siger) OF BRABANT, THOMAS AQUINAS, et al., were also condemned.

In the prologue to his letter of condemnation, the bishop relentlessly threatens anyone who teaches any of the 219 theses or fails to denounce their followers. He threatens excommunication and calls for denunciation and self-denunciation.

F. VAN STEENBERGHE (in: *'La Philosophie au XIIIe siècle',* 1991, p. 422) described the verdict as the most serious condemnation of the entire Middle Ages: *"la plus grave condamnation du moyen âge".* Nevertheless, the condemnation of 1277 did not prevent the Aristotelian writings from being read,[4] and their importance for university research and teaching increased. Nevertheless, the condemnation remained, for a long time, the determining factor for publicly held doctrinal opinions. We will further address this in the following discussion of the problematic concept of infinity.

6.3 The question of the existence of actual infinite quantities is critically examined

ARISTOTLE had claimed, in the third book of his *'Lecture on Physics'*, that the world is finite in its spatial extension and that there are no actual-infinite quantities in the natural real world. The finiteness of the world is, for ARISTOTLE, an empirically ascertainable fact, because, in his opinion, the whole space is enclosed within the visible vault of heaven.

ROGER BACON[5] (1214-1292) was of the opinion that the finiteness of the world could

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[4] BOETHIUS had already translated the Aristotelian writings on logic into Latin in the years 510-522. Further writings of ARISTOTLE were first translated into Latin by JACOB OF VENICE, WILLIAM OF MOERBEKE, GERHARD OF CREMONA, ROBERT GROSSETESTE and others in the 12th and 13th centuries. From the end of the 13th century onwards, all of ARISTOTLE'S writings (with the exception of the *'Eudemic Ethics'* and the *'Poetics'*) were accessible in Latin translations in Western Europe. In scholastic philosophy, ARISTOTLE became the dominant figure. - Cf. M. GRABMANN: *'Forschungen über die lateinischen Aristoteles-Übersetzungen des XIII. Jahrhunderts'.* Aschendorff'sche Buchhandlung Münster 1916.

The writings of PLATO only became known in Western Europe much later. The Platonic *'Timaios'* had already been translated into Latin by the New Platonist CHALKIDIOS in the 4th or 5th century, but only in the 12th century were the dialogues *'Menon'* and *'Phaidon'* translated (by HENRICUS ARISTIPPUS) into Latin. CHRYSOLORUS (died: 1415) had translated PLATO's *'Republic'.* The other Platonic writings only became known in the Western World from about 1450 onwards, when the Greek scholars fled from the advancing Ottomans, bringing the remaining ancient writings with them to the West. Constantinople was conquered by the Turks in 1453. In Western Europe, there began a renaissance of Platonism, which, however, also led to a fight against ARISTOTLE and scholastic philosophy. In 1459, the Mediceans founded a Platonic Academy in Florence, headed by MARSILIUS FICINUS (1433-1499). FICINUS translated many Platonic writings into Latin, all of which were printed in 1483. ANGELO POLIZIANO (1454-1494) translated PLATO's *'Charmides'* into Latin. The Platonic dialogues were not published in the original Greek until 1513, by ALDUS MANUTIUS in Venice.

[5] The English philosopher of science ROGER BACON was born in 1214, in Ilchester in the county of Somerset. He studied at the universities of Oxford and (from 1235 onwards) in Paris. He returned to England in 1252, and entered the Franciscan Order in Oxford in 1255. There, he began to work intensively on problems in physics. The other clergymen of his monastery saw only devilish magic in it and denounced him to the Pope in Rome. The general of the order JOHANNES FIDANZA, who called himself BONAVENTURA, brought about a ban on writing and the transfer into monastic custody. BACON began his imprisonment in Paris in 1257. He was only released after CLEMENT IV had ascended the

even be proven by logical and compelling evidence. In his *'Opus tertium'* (op. cit., Cap. 41, pp. 141-142), he argued as follows: If space extends to infinity everywhere, then a finite straight line *AB* could be extended beyond point *B* to infinity. The infinite straight line starting at point *B* could be put onto the infinite straight line starting at point *A*. According to EUCLID (*'Elements'*, Book I, 7th common notion), quantities that coincide are equal to each other. So, the two straight lines are equal to each other, although one is a proper part of the other - a contradiction! So, the universe cannot be infinite.

However, in 1277, the Bishop of Paris, STEPHAN TEMPIER, explicitly condemned such an opinion, as expressed by BACON and others. TEMPIER wrote:

> (LXXXVI) *"Prohibemus et... totaliter condemnamus: (...) Quod substantie separate sunt actu infinite. Infinitas enim non est impossibilis, nisi in rebus materialibus".* (No. 49 in Roland Hissette).
>
> [86: We completely forbid and condemn the statement that there is an actual infinite multitude of different spiritual beings, but infinity would not be possible in material things.]

TEMPIER wanted to say, among other things, that it would be wrong to claim that, in the natural material world, there can be, on principle, no actual infinite quantities. TEMPIER was convinced that God, in his omnipotence, could, if he wanted, create actual infinite quantities, even in the world of material things.

ROBERT GROSSETESTE (also called GREATHEAD, 1175-1253), for example, was willing to accept that there are actual-infinite sets, such as the set of all points on a line. He believed that, for us humans, this number was potentially infinite, but that, for a God, all the points on a line would form an actual infinite set. The number of points would be a certain number (*certus numerus*), but this number would be known only to God (cf. *R.* GROSSETESTE: *'Kommentar zur aristotelischen Physik, IV',* edited by R.C. DALES, p. 53). One feels reminded of the saying of the oracle of Delphi that only Apollo *knows the number of grains of sand* - cf. Chapter 5, section 5.4.

HENRY OF HARCLAY *(*ca. 1270-1317, chancellor of Oxford University from 1312 to 1317) also believed that God could overlook all the points of a line. (He thus affirmed the existence of actual infinite sets.) He even believed that God could also see the two points of a line that are closest to any given point, i.e., that are immediately adjacent. This view would, of course, solve ZENO*'s* paradox of the *runner in the stadium.* He did not realize, however, that a linearly ordered set of points could not be both *intrinsically densely ordered*

---

papal chair in 1265. CLEMENT IV asked BACON to explain his views. Thereupon, he wrote the work entitled *'Opus majus'* (the 4th part of this work is about 300 pages long and is entitled *'Specula mathematica'*). But NICOLAUS III, the successor of CLEMENT IV and GREGOR X, had BACON imprisoned again in 1279, and forbade the reading of his writings. The imprisonment lasted 10 years. He died on June 11, 1292, a few years after his release.

BACON became famous for his research in optics. He discovered that segments of glass spheres can have a magnifying effect. This led to the invention of spectacle lenses in the early 14th century. For BACON, the key to understanding nature was mathematics. He wrote, in his *'Opus Majus'* (Volume I, p. 106): *"Sola in mathematica est certitudo sine dubitatione"* [Only in mathematics is there certainty free from all doubts].

and *discretely ordered.* For more details, see the essays by JOHN E. MURDOCH cited at the end of this chapter, and also the contribution of JEAN CELEYRETTE in the volume *'Seventeenth-Century Indivisibles Revisited'* (V. JULLIEN, Editor), op. cit.

But can God completely execute the process of halving a given limited line, and then halving the two halves as well, and so on, *ad infinitum*? If He is omnipotent, then it should be possible for Him; but how would such a process take place?

ADAM WODEHAM, a disciple of WILLIAM OF OCKHAM, believed that God could not complete such a process of division, even if He had an eternity to do so,

"*si deus eternaliter divisisset.*"

For, if He could terminate this process of division at any time, then He should have started it (in retrospect) an infinite time ago; but then there could be no first halving step, said WODEHAM. - The model of this argumentation is again the *Zenonian* paradox of the *runner in the stadium*.

The Augustinian hermit GREGORY OF RIMINI (died: 1358), also a student of WILLIAM OF OCKHAM, objected to WODEHAM's assertion that the infinite process of division could take place in finite time. God could, at any time, perform the first halving step, halve the two halves again half an hour later, halve all four parts again a quarter of a hour later, halve all the pieces created so far an eighth of an hour later, and so on, *ad infinitum.*

6.4 BURIDAN's treatment of the infinity problem according to the method of '*sic et non*'

JOHANNES BURIDAN, another disciple of WILLIAM OF OCKHAM, was born in Flanders around 1292, and died in Paris around 1363. In his commentaries on the two Aristotelian writings *'Lectures on Physics'* and *'De Caelo'*, he dealt in detail with the problem of whether the existence of infinite magnitudes is possible or not.

Here, he applied the usual scholastic method of **sic et non** ("so and not so"). This is a method of deciding a controversial question by collecting the opposing opinions of the authorities from the literature, first the arguments in favor of a particular assertion and then the arguments against it. Once the conflicting opinions have been collected in this way, a final consideration must either resolve the contradiction or extract what is undisputed. This method goes back to ABELARD (1079-1142).

BURIDAN treats the *Quaestio "Utrum possibile sit esse corpus infinitum"* (i.e., whether it might be possible that infinite objects exist) in the scholastic method of **sic et non**, both in the 1st book of the *'Quaestiones in Aristotelis De Caelo et Mundo',* Quaestio 17 (pages 317-323 in the edition of BENOIT PATAR, op. cit.), and also[6] in his Commentary on the Aristotelian Physics, book 3, Quaestio 19 (pages 63v-65v, op. cit.).

(i) BURIDAN first presents ('**arguitur quod sic**') some positive arguments, including the argument by GREGORY OF RIMINI in which it is claimed that Infinity is possible in material things.

[6] The page references in the Paris print from 1509 are not always correct. For example, page 63 erroneously bears the number 62.

It is possible that God, in His omnipotence, can actually create infinitely many things. One can imagine that He could create some object (e.g., a stone) in half an hour, and a second stone of the same size in the next quarter of an hour, and the same such stone again in the next eighth of an hour again, and so on, *ad infinitum*. The division of the hour into infinitely many parts would make it plausible that God could, if He wanted, create an infinite number of objects in a single hour, and thus, altogether, an infinite multitude,

> *"... cum infinite sint medietates proportionales hore sequitur quod in fine hore essent infiniti lapides pedales. ... Ergo magnitudinem infinitam potest deus facere."*
>
> [... and just as a single hour has infinitely many proportional parts, so at the end of an hour there are infinitely many one-foot stone blocks. ... Thus, God would be able to make something which is infinitely large.]
>
> J. BURIDAN, Quaestio 19 on Aristotelian physics, page 63verso, 2nd column, op. cit..

(ii) BURIDAN then deals (in the section '**arguitur quod non**') with a number of objections. BURIDAN*'s first objection* is the following:

> *"Ideo contra istam imaginationem ego volo probare quod, quamvis in qualibet medietate proportionali unius horae possit Deus facere lapidem pedalem, tamen impossibile est quod in qualibet medietate proportionali illius horae faciat lapidem pedalem, quoniam hoc implicat contradictionem, ut probabo.*
> *Et est prima probatio. ... Modo ultra ponamus quod Deus, quo ordine fecit illos lapides in una hora, e converso ordine posset illos destruere in una alia hora; et tunc nunquam destrueret duos lapides simul; ... Et sic esset dare unum lapidem primo destructum, et ille non esset nisi qui ultimo fuit factus. Et sic habeo propositum.* (Pages 319-320 in the edition provided by B. PATAR, op. cit.)
>
> [Therefore I want to prove against this thought-experiment that although God could possibly create a one-foot stone in *each* arbitrary proportional part of an hour, it is nevertheless impossible that He will (actually) create a one-foot stone in *all* proportional parts of an hour, because this implies a contradiction, as I will prove.
> Here is the first proof. ... Let us assume, finally, that God, just as He created the orderly row of stones in one hour, He can also destroy all the stones in reverse order again in one hour, and that He never destroys two stones at once. ... Hence, one has to admit that there has to exists one stone which will be destroyed first. (However, such a first stone does not exist.) And in this way I have provided the proof.]

I have omitted, in the rendering of the text, that part in which BURIDAN shows that, in the process of creating the infinite sequence of stones, there is a first created stone and a second created stone, etc., but that there is obviously no last created stone. The end of the proof is truly remarkable: BURIDAN says that one can impute to God with equal right that He can undo everything He created in the one hour. Thus, BURIDAN takes the word 'undo' both in the sense of 'destroying what was created before' and in the sense of reversing the order.

This counter-argument apparently reproduces ZENO*'s* paradox of the *runner in the stadium* (see above). The assertion that, if an omnipotent being is to destroy a series of objects, it must begin with the destruction of a *first* object seems to be evident. But the evidence comes from experience in dealing with finite sets of objects, and the wrapping of

paradox in processes of creating and destroying objects from the natural real world suggests that we should draw on this experience. However, the laws that apply to finite sets need not apply to infinite sets as well. Therefore, both BURIDAN and ZENO are in error when they claim that, for reasons of rationality, the above-mentioned process cannot be reversed. However, BURIDAN has a plethora of further objections. The following objection is selected from BURIDAN's work *'Quaestiones in Aristotelis De Caelo'* (liber I, Quaestio 17).

BURIDAN*'s second objection*: If the above argument were valid and God could create an actual infinite set of one-foot stones after one hour, then one could just as rightly have God divide a given finite straight line in the course of an hour into infinitely many parts as follows. In the first half hour, he cuts the given straight line in the middle into two equal parts. In the next quarter of an hour, he again cuts each of the two sections in the middle into two smaller sections of equal size. In the next eighth of an hour, he cuts each of the four sections in the middle into two sections of equal size, etc., *ad infinitum*. At the end of an hour, the given straight line would be divided into an infinite number of disjoint parts. If, after one hour has elapsed, the process of decomposition were to be completed as a whole, then the given straight line would be the union of all the infinitely many sections. These sections would all have the length 0, and would therefore be points. The given straight line would therefore be a totality of points. But, by juxtaposing points, a line is never created, as ARISTOTLE showed in his *'Physics Lecture'* (Book VI, Section 1, 231a24-26). (See also the writing attributed to ARISTOTLE*:* Περὶ ἀτόμων γραμμῶν (*De Lineis Insecabilibus, On indivisible lines*)). The contradiction shows that such an infinite process can never be thought of as finished.

In this counter-argument, the two Zenonian paradoxes are connected, that of *Achilles and the turtle* and that of the *runner in the stadium.* Division points are now introduced in all previously constructed sections. The error is the same as in the paradoxes concerning motion. It is pretended that the two processes, increasing the number of parts and decreasing the sizes of the parts, can be considered independent of each other. But the two processes are interdependent, and, after only a finite number of steps, one has finitely many parts, the sum of which is the initial given object. This fact remains valid throughout the whole process. The fact that, by *'juxtaposing'* points, a line is never created is true for finitely many points, but, when 'all' points are taken together, the line remains in its totality.

BURIDAN*'s third objection*: In Quaestio 19 of the 3rd book of his commentary on the Aristotelian *'Lecture on Physics'* (op. cit., p. 64 recto), BURIDAN also gives the following counterargument. If one concedes to God that He can create, in the proportional parts of an hour, an actual-infinite sequence of objects of equal size, then He could analogously create, in the proportional parts of an hour, an actual-infinite square as follows. He starts with any square, the side length of about one foot. In the first half hour, He places a "gnomon" (i.e., a right-angled hook) on this square, so that a square of the side length of two feet is created. In the next quarter of an hour, He places an even larger right-angled hook on this square, so that the resulting square has a side length of 3 feet. In the next eighth of an hour, He enlarges it to a square of side length 4 feet, and so on, *ad infinitum*. God would thus have created a square of infinite side length after one hour. The diagonal of the square would

also be infinite. But the side and the diagonal can be laid on top of each other in such a way that they coincide, and are therefore of equal length (BURIDAN presumably assumes that the seventh 'common notion' from the *'Elements'* of EUCLID , Book I, also applies in the present situation). Consequently, the square over the side would be equal to the square over the diagonal. But, as already proved in the Platonic *'Menon'*, the square over the diagonal is twice as large as the square over the side - a contradiction!

BURIDAN gives further objections, but we do not want to deal with all of them. He treats all of these problems in a *quaestio*, which asks whether the existence of an actual-infinite magnitude is possible:

"*... utrum possibile sit esse corpus infinitum."*

(iii) **Resolving the contradictions**. After the arguments *'for'* (**sic**) and *'against'* (**et non**) are presented, BURIDAN tries to resolve the contradictions in a final response. Among his arguments is the following linguistic analysis.

BURIDAN accepts the argument by GREGORY OF RIMINI that God can possibly create, at any time, an object of the kind discussed. One could therefore speak of an infinite number of created objects in the *syncategorematic* sense (i.e., in the *distributive sense* of finitely many objects in an arbitrarily large number). But BURIDAN adds that the first of the above-mentioned objections show that one cannot speak of an infinite number of created objects in the *categorematic* sense (i.e., in the *collective sense* of an actual infinite totality of created objects). Notice that, when speaking of *'all'* created stones, this means, in the *syncategorematic* or *distributive sense*, simply *'qualibet'* (i.e., *'each'* or *'every one'*). However, speaking of *'all'* created stones in the *categorematic* or *collective* sense means *'altogether'* (i.e., *'omnes'* or *'cuncti'*). Here, *'cunctus'* is derived from *'conjunctus'* and more precisely means *'all things unified somehow to a whole'*. Thus, BURIDAN accepts that, in the argument of GREGORY OF RIMINI, one can speak of *'all stones'* in the distributive sense, i.e., of stones created *'in each (single) arbitrary part of an hour',* but not in the collective sense of an actual infinite totality of stones created in an hour altogether. Thus, according to BURIDAN, GREGORY'S argument is convincing, provided his use of the words "all stones" is understood in the syncategorematic sense, and not in the categorematic sense.

In this way, the above-mentioned *'Questio'* was discussed by BURIDAN in detail. Applying the method of *sic et non*, he settled the contradictions between the opposing arguments. He tried to support the idea that, for reasons of rationality, it is impossible, even for a deity, to create an infinite magnitude in the real world, thus contradicting one of the theses of TEMPIER (thesis No. 86). However, BURIDAN's 'reasons of rationality' have all been modeled after the Zenonian paradoxes, and therefore cannot be considered as truly convincing. At BURIDAN's time, 'infinity' was still an enigma.

The question of whether there are actual-infinite multitudes of natural real things was raised again in the 19th century. BERNARD BOLZANO (1781-1848) tried to give an answer in his booklet *'Paradoxien des Unendlichen'* (op. cit., Leipzig, 1851, § 25). He took the existence of a God for granted and concluded from the usual definition of the concept of a deity that such a God

"... unendlich Vieles (nämlich das All der Wahrheiten) weiß, unendlich Vieles will, und

daß er Alles, was er will, durch seine Kraft, nach Außen zu wirken, in Wirklichkeit setze."

[... knows infinitely much, namely the totality of all truths, wants infinitely much. and that He makes all that He wants come true through His power to act outwards.]

BOLZANO believed that he could conclude from this concept of a deity that the quantity of all created things in the world is infinite.

However, such a 'proof' was no longer convincing in the 19th century. It is noteworthy that RICHARD DEDEKIND (1831-1916) also argued, a little later, that his own *'world of thoughts, i.e. the totality S of all things which can possibly be the object of* [his] *thinking'* is infinite. In fact, to each thought, the 'thought on this thought' is another thought, and the 'thought on the thought, which is a thought on the (first) thought', is again another thought, etc. DEDEKIND claimed that his *'world of thoughts which can possibly be the object of* [his] *thinking*' is an 'infinite system' (cf. statement 66 in his writing *'Was sind und was sollen die Zahlen?'*, 1888. See also Chapter 13 and Chapter 19).

6.5 Concluding remarks

The concept of infinity was vividly discussed in antiquity and in the Middle Ages. But the enormous difficulties associated with this concept could not be resolved. The concept of infinity remained dark and blurred, and thus could not find its way into mathematics. But all of these medieval discussions contributed to the many difficulties surfacing in the first place, and thus to their ability to be clarified in the course of time (cf. e.g., the book *'Seventeenth-Century Indivisibles Revisited',* edited by VINCENT JULLIEN, op. cit., and other works on the history of Mathematics). Slowly, the timidity in dealing with the concept of infinity disappeared.

The 'invasion' of the infinite into mathematics happened as early as the early 17th century,

- first, in *projective geometry* (GIRARD DESARGUES, 1638), with the introduction of straight lines of infinite length,
- then, in the *Calculus of Fluxions* (ISAAC NEWTON, 1772/1687/1704/1736) and the *Differential- and Integral-Calculus* (GOTTFRIED WILHELM LEIBNIZ, 1675/1684), with the introduction of (potentially)-infinitely-small quantities,
- then, in the '*Infinitär Kalkül*' (PAUL DU BOIS-REYMOND, op. cit., in spring 1873), with the discovery that there are, in the set of all divergent positive real functions with respect to their final course, *uncountably* (!) many degrees of becoming infinite,
- and finally, in *set theory* (GEORG CANTOR, December 1873), in which the existence of infinite sets of different cardinalities was discovered.

It should be emphasized that the concept of infinity could only be introduced into mathematics after all cosmological and theological references had been removed. Whether there are things of infinite number or infinitely large things in the natural real world, or whether an omnipotent God could simultaneously recognize an infinite number of points on a straight line, is not a question that mathematics is concerned with.

In Chapter 3, we came to a comparable result. There, the Aristotelian approach to

geometry failed because of the all too close connection of geometric objects with the objects of the real world. We now see that a mathematical treatment of the concept of infinity must also fail if it is too closely overloaded with cosmological or theological problems. For a logical-mathematical analysis of the concept of infinity to be possible, the concept must be freed not only from all *"Erdenrest"* (*earthly residue*), but also from all theological ties.

Mathematics is only about '*frameworks of concepts*', as DAVID HILBERT repeatedly expressed it (cf. Chapter 20). However, this insight only became generally accepted in the course of the 19th century. In the late 19th century, it became possible to treat the concept of infinity in an abstract form on the basis of a new theory, set theory.

For the history of mathematics, the discussion of ZENO*'s* paradoxes in antiquity, in the medieval period, and in modern times is also of great importance, because it has helped to clarify the concept of the continuum, and thereby the concept of real numbers. This term is one of the fundamental concepts of mathematics. The clarification only gradually succeeded in the 19th century.

In earlier times, VOLTAIRE*'s* advice might have been good advice, to think of one' s own health, rather than struggling with the infinite. Today, the concept is so well developed that dealing with infinite quantities in mathematics no longer poses any danger.

However, we must be careful and must not so simply claim that all dangers have been eliminated, for the dangers are only really eliminated when there is a proof of consistency for axiomatic set theory. But such a proof is not possible according to the GÖDEL-BERNAYS (1931/1939) incompleteness theorem. - *One cannot completely safeguard and secure mathematical thinking.*

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Part II

Philosophy of Mathematics in the 16th, 17th and 18th Century

In the sixteenth century, the question of the status of Euclidean geometry among the sciences was discussed with great eloquence and expertise. There was widespread agreement that it did not meet the Aristotelian requirements for a science, but nevertheless had the highest degree of clarity and certainty of all disciplines. We report on this debate in Chapter 7.

In this debate mathematics has been repeatedly praised for its *apodictic certainty.* Yet 'apodictic certainty' is a certainty supported by *irrefutable and unconditional demonstrations.* The question whether the proofs in mathematics are really *completely unconditional* has been controversially discussed since the 16th century. Finally, the somewhat more precise question was asked as to how great the scope of our knowledge *a priori* really is.

In Chapter 8 we discuss the so-called *'nativism*' that RENÉ DESCARTES and others have advocated. Here the view is taken that knowledge of mathematical truths is based on the '*clear and distinct*' intellectual view of the *ideas that are innate* to us (nativus = innate). According to this view, the scope of our knowledge *a priori* is very large.

In Chapter 9 we discuss the view of the 'enlightened' JOHN LOCKE, who vehemently opposed Cartesian nativism and therefore considered the scope of our mathematical knowledge a priori as being rather small.

Chapter 10 deals with '*rationalism*'*,* according to which all the truths of arithmetic and geometry can be drawn from the human *ratio* alone (THOMAS HOBBES, GOTTFRIED WILHELM LEIBNIZ et al.).

In Chapter 11 we discuss the '*empiricism*' of DAVID HUME, GEORGE BERKELEY, JOHN STUART MILL and others, according to which there is no knowledge *a priori* in mathematics.

In Chapter 12 we come to the '*critical philosophy*' *of* IMMANUEL KANT and his efforts to redefine the scope of our *a priori* knowledge.

Chapter 7
On Certainty in Mathematics

"Nihil certi habemus in nostra scientia, nisi nostram mathematicam." [In all our knowledge there is no certainty except in our Mathematics.]
NICOLAUS CUSANUS: *'Trialogus de possest',* 44,1-2.

"Je me plaisois surtout aux Mathématiques, a cause de la certude & de l'evidence de leurs raisons."
[I especially like mathematics because of the certainty and evidence of their reasons.]
RENE DESCARTES: *'Discours de la Méthode' (1637).*

The question of the status of Euclidean geometry among the sciences was the subject of an intensive debate in the 16th century, the age of the Renaissance. There was widespread agreement that Euclidean geometry did not meet the Aristotelian requirements for a science, but nevertheless had the highest degree of *clarity and certainty* of all disciplines.

But does Euclidean geometry also lead to *apodictic certainty,* i.e., to a certainty that rests upon 'scientific proof' in the sense of ARISTOTLE? Do we have to subordinate the *Euclidean axiomatic method* under the *Aristotelian apodeixis*, or is the Euclidean axiomatic method superior to the Aristotelian apodeixis?

In this chapter, we will report on the discussion of this controversial issue in the 16th century. We begin with a description of how it came about in the first place that *Euclidean deduction* was contrasted with *Aristotelian apodeixis.*

7.1 The publication of the works of EUCLID and PROCLUS in the original Greek

In Latin-speaking Western Europe, Greek-language mathematical works had already fallen into oblivion in the early Middle Ages. In the 10th century, for example, not a single complete text of EUCLID's *'Elements'* was available in that part of the world. Only the Latin translation from Greek, which BOËTHIUS (ca. 480-524) had made shortly after 500 C.E., was still available in short excerpts.

It was not until the 12th century that interest in mathematical treatises awakened there again. To familiarize oneself with the *'Elements'* of EUCLID, for example, was possible only after the realization of complete Latin translations from Arabic [ADELARD OF BATH (ca.

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1120), GHERARD OF CREMONA (ca. 1150) and CAMPANUS OF NOVARA (ca. 1255)]. A complete Latin translation from Greek, carried out in Palermo around 1160, became known only to a very small circle of scholars.

Only after 1453, when the Turkish Emir MEHMET II conquered Constantinople, and, in consequence of this disaster, all scholars of the imperial court, and many of the monks as well, fled to Italy and took the old manuscripts of the Greek philosophers, poets and scientists from their libraries and monasteries with them, did many Greek-language mathematical works reach the western world. Among those works was the original Greek version of EUCLID's *'Elements'*.

Thus, it came about that, in Western Europe, numerous works that were not known at all or only in Latin translations from Arabic finally became known again in the Greek original (cf. Chapter 6, footnote 4). The commentary by PROCLUS (410-485) on the first book of the Euclidean *'Elements'* had not been translated into Arabic, and only then was reintroduced in the Western world. In particular, the publication of many Platonic writings caused real enthusiasm. A renaissance of *Platonism* began, which also led to a fight against ARISTOTLE and scholasticism.

The original Greek version of the Euclidean *'Elements'* was printed for the first time, together with PROCLUS' commentary, in Basel in 1533, in the printing-office of JOHANNES HERWAGEN. The editor was SIMON GRYNAEUS (SIMON GRÜNER, 1493-1541). A Latin translation of PROCLUS' commentary was prepared somewhat later by FRANZISCUS BAROCCIUS (F. BAROZZI, 1537-1604) and published in Padua in 1560.

GRYNAEUS preceded his edition of the Euclidean *'Elements'* with a foreword (Praefatiuncula) of 8 pages. In it, he first complained of the poor teaching in all subjects, which was conducted in wild confusion without any predetermined order and required the students to agree to every assertion made without their truth being tested. He recommended the *'Elements'* of EUCLID as a textbook of arithmetic and geometry, because, in the *'Elements'*, the material was presented convincingly and in well thought-out order, and each assertion was supported by a clearly formulated proof. For GRYNAEUS, axiomatically-constructed geometry was the perfect standard of the scientific method:

"... Geometriae, quae methodi totius absoluta et perfecta formula est."

Therewith, GRYNAEUS called into question the leading role that Aristotelian scientific doctrine still played at the time. The method that ARISTOTLE had worked out in his *'Second Analytics'* was still considered as the perfect standard of the scientific method. It was the undisputed ideal of all disciplines. Even theology was developed by some scholastics according to the Aristotelian model, and the articles of faith (*articuli fidei*) were not treated as *postulates*, but declared as *true principles* of theology (cf. HERMANN SCHÜLING, op. cit.).

The leading role of the Aristotelian scientific doctrine resulted from its device of basing proofs only on *'reasons'* (causes), namely, the four types of cause mentioned in Chapter 3. This means that it is not sufficient only to show *that* the alleged facts are the consequences of other statements that are already accepted, but that it is necessary to prove *why* they are true. The distinction between these two types of proof is connected with the expressions *'hoti'* (ὁτι, *quod*) and *'dihoti'* (διότι, *quare, cur*) - see Chapter 3, section 3.2.

In contrast to the Aristotelian method of proof, the Euclidean method of proof is a device

that derives its theorems simply from *postulates* (see Chapter 4), i.e., from statements that had to be accepted (without proof) in advance. Thus, the Aristotelian conception of a scientific theory seems to be much more pretentious than the Euclidean conception.

The Euclidean postulates are neither true nor false, since they do not refer to sensible objects of the real world. They result from sense-perception through *abstraction and idealization!* However, proofs from postulates in Euclidean geometry have a much higher degree of *clarity and reliability* than any proof in a scientific theory in the sense of ARISTOTLE.

An intensive discussion of the differences between the two forms of scientific theory and their respective advantages and disadvantages began in the 16th century. We will report on this discussion in the following sections.

7.2 The differences between Aristotelian *'apodeixis'* and Euclidean *'demonstration'*

As we have elucidated in Chapter 3, section 3.2, among the four Aristotelian causes, only the *causa formalis* is relevant in mathematics. In it, the *'essence'* (εἶδος καὶ τὸ παράδειγμα) of the objects belonging to a certain species is given. [This kind of *cause* is called *'formal'*, since the Greek word εἶδος is usually translated into Latin as *'figura'*, or *'forma'*, and hence *'causa formalis'* refers to a *cause* in which the (spiritually perceptible) *form* of an object is exposed.] The Aristotelian requirement for a mathematical theory to be a scientific theory, hence, means that all its theorems need to have proofs based on the causa formalis alone. These proofs have to make clear *'why'* (διότι, *quare, cur*) it behaves this way or that way, as is claimed in the formulations of the theorems. Notice also that, according to ARISTOTLE, proofs always have to be performed syllogistically.[1]

Thus, we may say, in a slightly simplified way, that the Aristotelian requirement for a mathematical theory to be a scientific theory is that all of its theorems have proofs that are based solely on assertions on the existence and the essence of the objects under consideration.

The effect of this requirement was enormous. Almost all philosophers in the Middle Ages accepted this requirement and supported and defended it in their writings. THOMAS AQUINAS (1225-1274), for example, in his *Commentary on the Trinity Tract* of BOETHIUS (*'Expositio super librum Boethii De trinitate',* Quaestio VI, Articulus I, Responsio), written in Paris between 1255 and 1259, described the Aristotelian view of the scientific character of mathematics as follows:

> *"In scientiis enim mathematicis proceditur per ea tantum, quae sunt de essentia rei, cum demonstrent solum per causam formalem."*
>
> [For in the mathematical sciences progress is made only by what belongs to the essence of things, because they (the mathematicians) prove only by the *causa formalis*.]

In scholastic commentaries, proofs that only establish *'that'* (ὅτι, *quod*) the alleged fact holds true were called *demonstrationes quia*, and proofs that go so far as to establish *'why'* (διότι, *cur, quare*) the alleged fact holds true were called *demonstrationes propter quid.*

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[1] This, however, is not absolutely clear. We refer to JONATHAN BARNES: *'Proof and the Syllogism'* (1981, op. cit.), section IV, p. 29.

The question now is whether Euclidean geometry also satisfies the Aristotelian requirements for a scientific theory. Are all demonstrations in the *'Elements' demonstrationes propter quid?*

Geometry is presented in the Euclidean *'Elements'* as a corpus of theorems and problems that are proved, resp. solved, on the basis of *definitions, postulates* and a few more or less evident *common notions* (see Chapter 4). These definitions, however, are not meant to contain the essence (in the Aristotelian sense) of the defined objects, and the proofs are not given in syllogistic form.

Notice also that the Aristotelian and Euclidean conceptions of geometry already differ in the domains of objects that are to be investigated. The objects of an Aristotelian-conceived geometry must have a material carrier, i.e., they must be present in the real world *'in the manner of Hyle',* and result from an *abstracting* process. In contrast, the objects of Euclidean geometry are considered as objects that are obtained through *idealization* and are seizable only in the form of concepts. This indicates that geometry, as carried out in EUCLID*'s 'Elements',* should be considered perhaps more as an *art* (ars, τέχνη) than as a *science* (scientia, ἐπιστήμη). We will come back to this question later.

7.3 The dispute over the question as to whether Euclidean geometry is a science in the Aristotelian sense, or not[2]

ALESSANDRO PÍCCOLOMÍNI (1508-1578) was the first to thoroughly examine the traditional view. He argued (1547, op. cit.) that the proofs in mathematics are *not demonstrationes propter quid,* because they are not based on causes, i.e., not inferred from definitions of essence. Thus, according to PÍCCOLOMÍNI, the mathematical disciplines would not be sciences in the Aristotelian sense.

PÍCCOLOMÍNI, however, added that mathematics nevertheless had the highest degree of clarity and certainty. He based his opinion on the fact that mathematical objects were created by the human mind, and that the human mind can make more precise statements about the things it has created than about objects that it finds in nature outside of its own thinking.[3] The conviction that the human spirit is the creator of mathematical things had

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[2] In the following dispute, a few special terms, which ARISTOTLE needed to discriminate properties that may belong to an object, will be applied. ARISTOTLE discriminated among these properties with respect to the strength of the binding of the property upon the object.

(a) With the term *Essential* those properties of an object are described that indicate the species and the next higher genus to which the object under discussion belongs (cf. Chap. 3, section 3.1).

(b) A property is an *attribute* of an object if it belongs *'de facto'* to the object, that is, if it belongs *kath' hauto* (καθ' αὑτό, *per se*) to the object. *Attributes* are, hence, those properties of an object that cannot be denied without destroying the object. Notice that all essential properties are attributes.

(c) A property belongs *accidentally* to an object if it belongs only temporarily, or incidentally, or by chance, or occasionally to it, that is, if it belongs *'kata symbebêkos'* (κατὰ συμβεβηκός, *per accidens*) to the object. (In Greek, συμβαίνειν means 'to accompany', and τὸ συμβεβηκός is the subordinated property.)

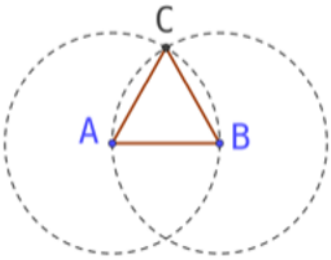
[3] RENÉ DESCARTES (cf. Chapter 8, footnote 5) and KURT GÖDEL (cf. Chapter 16, proof of (†)) have also expressed this idea.

been expressed before him, including by PROCLUS, ROGER BACON, NICOLAUS CUSANUS[4] and others. GERONIMO CARDANO (1501-1576), in his great book on natural philosophy *'De subtilitate, libri XXI'* (Nürnberg, 1550), supported this view and said that it is therefore not surprising that geometry is the most subtle (i.e., the most exact and certain) of all sciences: *"Nihil mirum igitur Geometriam esse omnium scientiarum subtilissimam"*.

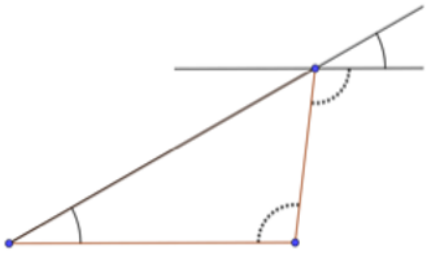
The Spanish scholar BENEDICTUS PERERIUS (PERERA, PEREYRA: 1535/1537-1610), who taught philosophy and theology at the Collegio Romano in Rome, also examined, in his work *'De Communibus omnium rerum naturalium Principiis & Affectionibus'* (Rome 1576, Lyon 1585, op. cit.), the question of whether mathematics, physics, etc., are sciences in the Aristotelian sense. He too (like PÍCCOLOMÍNI and others before him) came to the conclusion that the mathematical disciplines, arithmetic and geometry, were not true sciences (in the Aristotelian sense).

As 'proof', he showed (in the third book of the above-mentioned work, Cap. III, p. 114 ff.) that mathematics does not deal with any of the four Aristotelian types of cause. In the introduction, he discussed the problem **I**,1 from the *'Elements'* of EUCLID, in which an equilateral triangle is to be constructed over a straight line connecting two arbitrarily given points *A* and *B* (we discussed this in Chapter 4, pp. 60-61).

PERERIUS argued that the equality of the three sides is not inferred from any 'cause' (in the Aristotelian sense). The equality of the three sides only results from the fact that the sides of the triangle in this diagram are also radii of circles. This is a property that belongs to the sides only *per accidens* (κατὰ συμβεβηκός), and is not an essential property of straight lines. He concluded that this proof is certainly not a *demonstratio propter quid*.



EUCLID, *'Elements'*, **I**,1



EUCLID, *'Elements'*, **I**,32

PERERIUS also gave an examination of the proof of the theorem about the sum of interior angles in a triangle, the so-called *Triangle-Sum-Theorem* (EUCLID, *'Elements'*, **I**,32). PERERIUS stated (op. cit., p. 40-41) that the proof depends on the construction of a line

[4] PROCLUS wrote, in the first preface to his *commentary on* EUCLID, *'that the human mind* is *the creator of the mathematical forms of being and concepts'*. NIKOLAUS VON KUES (1401-1464) wrote, about a thousand years later, in his essay *'De Beryllo'* from 1458 (there in Chapter 33), that it is our mind that produces the mathematical things: *"ut mentem nostrum, quae mathematicalia fabricat"*, and that the mathematical things, even if they are obtained through abstraction, are truer in our human mind than in our senses: *"mathematicalia, quae a sensibus abstrahuntur, vidit veriora in mente."*

drawn parallel to one side of the triangle and through the point of the triangle opposite to that side. The (unique) existence and essence of the parallel lines is not a logical consequence of that - and only that - which is contained in the *definitio essentialis* of the concept of a triangle. The (unique) existence of the parallel line is guaranteed only by a postulate, namely, the Euclidean 5th postulate. Thus, the proof of the *Triangle-Sum-Theorem* is not based on the causa formalis. PERERIUS concluded that this example again shows that, in mathematics, most proofs are not *demonstrationes propter quid.*

PERERIUS used many other arguments to support his view that mathematics is not a science (in the Aristotelian sense):

> *"Mea opinio est, Mathematicas disciplinas non esse proprie scientias",*

it says on page 40 (liber primus, Cap. XII). He summarizes his thoughts as follows:

> *"Scire est rem per caussam cognoscere propter quam res est; & scientia est demonstrationis effectus: demonstratio autem (loquor de perfectissimo demonstrationis genere) constare debet ex his quae sunt per se, & propria eius quod demonstratur; quae verò sunt per accidens, & communia, excluduntur à perfectis demonstrationibus: sed Mathematicus neque considerat essentiam quantitatis, neq; affectiones eius tractat prout manant ex tali essentia, neque declarat eas per proprias caussas, propter quas insunt quantitati, neque conficit demonstrationes suas ex praedicatis propriis, & per se; sed ex communibus, & per accidens, ergo doctrina Mathematica non est proprie scientia."*
>
> PERERIUS 1585, op. cit., liber primus, Cap. XII, p. 40.
>
> [Knowledge is to know a thing by the cause why it exists, and science is that which is produced by a proof. A proof (and I am speaking here of the most perfect kind of proof) must depend only on those things that exist *per se* and that belong to what is being proved. All that which is only *accidental* and *general* is excluded from *perfect proofs*. But a mathematician does not consider the *essence* of a quantity [to be examined], nor does he treat the *affections* that are an outflow from it, nor does he explain them by the peculiar causes by which they are present in quantity, nor does he make his proofs with real and peculiar properties, but only by means of *general and accidental properties*. So mathematics is not a real science].

MARTINUS SMIGLECIUS (MARCIN SMIGLECKI, 1564-1618), a Polish philosopher, theologian and logician, took up the above-mentioned thoughts of PERERIUS in his *'Logica'* (Ingolstadt, 1618). Just like PERERIUS, he emphasizes that, in mathematics, conclusions are not drawn from causes. In particular, for example, in the problem **I**,1 from the *'Elements'* of EUCLID, conclusions are not drawn from the definition of the essence of the discussed triangle, but from the relationship of the triangle to the larger, more comprehensive figure. SMIGLECIUS writes (we quote from SCHÜLING, op. cit., pp. 51 and 135),

> *"...in Mathematicis non probantur proprietates ex essentia subjecti, sed ex habitudine ad aliam figuram: Ergo non probantur per veram causam essendi."* (SMIGLECIUS, op. cit., p. 306).
>
> [In mathematics, properties are not derived from the essence of the subject, but from its relationship to another figure. Thus it is not proved from true reasons of being.]

This is a very good remark, which, at the same time, makes clear that the answer to the

question as to whether or not a given geometric proof is inferred from *causes* depends on the diagrams in which the given figure is embedded. Proofs in geometry can be executed only under inclusion of the texture and structure of all points and all lines of the entire surrounding plane or space. (Notice that such an embedding of a given figure in a sufficiently rich, structured geometric plane is a fundamental method in Euclidean, pre-Euclidean and post-Euclidean geometry, but was unfortunately overlooked by ARISTOTLE - cf. Chapters 3 & 4.)

JOSEPH BLANCANUS (1566-1624), a disciple of the mathematician CHRISTOPH CLAVIUS (CHRISTOPH SCHLÜSSEL, from Bamberg, 1538-1612), wrote an extensive treatise on the problems discussed here. This treatise appeared in 1615 as an appendix to his work *'Aristotelis loca mathematica'*. BLANCANUS joins PROCLUS and many others in wanting to show that Euclidean geometry falls under the Aristotelian apodeixis. In Chapter 2 of his treatise, he discusses the problem **I**,1 from the *'Elements'* of EUCLID, which PERERIUS had already discussed (see above). But, while PERERIUS wanted to show that the solution of the problem only comes about *per accidens* and not by giving causes, BLANCANUS, on the contrary, wants to show that the solution can be found through causes. To do this, however, he must reformulate the problem a little:

> *If a line AB is given, then the triangle A,B,C, where C is one of the two intersections of the two circles with radius AB around point A, resp. around point B, is an equilateral triangle.*

This reformulation allows BLANCANUS to infer the equality of the three lines *AB, BC* and *CA* from the usual definition of the essence of a circle. According to ARISTOTLE, such a conclusion results from the *causa formalis*.

That is correct, but it must be noted that BLANCANUS speaks of a different configuration than EUCLID. Moreover, BLANCANUS has messed up the punch line, because the 'wit' of EUCLID*'s* approach lies in constructing the equilateral triangle by drawing two circles. The introduction of the two circles is the creative thought. BLANCANUS ignores this thought by assuming that the two circles are already given. On the other hand, BLANCANUS has, by all measures, hit the central nerve of the Aristotelian-conceived geometry with his remark, which is only feasible if, in all discussions of diagrams, all necessary auxiliary lines are always already present and existent (and obtained through abstraction).

We may add that many other philosophers intervened in the dispute as well, and some of them rather passionately, furiously, roughly and rudely, e.g., SIMON SIMONIUS, JACOB SCHEGK (1573, op. cit.) and a few others. For those who would like to learn more about the views of the people participating in the dispute, we refer to the carefully researched books by HERMANN SCHÜLING 1969, op. cit., and PAOLO MANCOSU 1996, op. cit.

7.4 ARISTOTLE's own argument

It is perhaps a bit surprising that no mathematician or philosopher in the 16th century made a reference to ARISTOTLE's own argument concerning the problem as to whether proofs in classical (i.e., Euclidean, pre-Euclidean & post-Euclidean) geometry are demonstrationes propter quid. In fact, ARISTOTLE himself showed, in his *'Metaphysics'*, V, §30, 1025a30-34, that the proof of the so-called *Triangle-Sum-Theorem* (EUCLID, *'Elements',* **I**,32), as it

was given already by the Pythagoreans (cf. PROCLUS, book 4 of his *'Commentary on* EUCLID*'*), is not based on a *cause*. To show this, ARISTOTLE discusses a property of geometric figures, namely, the following property of

(*) *"having the sum of their interior angles equal to two right angles"*.

The Pythagoreans (cf. EUCLID*,* in his *'Elements',* **I**.32) proved that this property (*) belongs to all triangles. ARISTOTLE concluded that the property (*), hence, is a property that all triangles do have eternally and that (*) is, *'de facto'*, a property of all triangles. Thus, (*) is a property that belongs to a triangle *per se*. But this property *does not belong to the essence of a triangle*, since, in the *definition of the essence* of a triangle, it is only mentioned that it belongs to the genus of plane figures, and moreover consists of three non-collinear points and three distinct straight lines between these points. This is all that is given in its definition of essence.

In the terminology of ARISTOTLE, the property (*) is συμβεβηκὸς καθ' αὑτό, thus *'per accidens'* (κατὰ συμβεβηκός), as well as *'per se'* (καθ' αὑτό). In fact, (*) is not an *essential* property of a triangle, as was indicated above. But (*) is a property of a triangle that falls to its share from the exterior texture and structure of the geometry of which the triangle is a part, and hence is κατὰ συμβεβηκός (i.e., *per accidens*). Since (*) expresses a *proprium* (ἴδιον) of a triangle, the property expressed in (*) belongs *'per se'* (καθ' αὑτό) to it. (Cf. WOLFGANG KULLMANN, 1981, op. cit., for a further discussion of the problems that show up here.) We may conclude from this discussion that ARISTOTLE's own opinion was that geometry, as carried out by the Pythagoreans (and also later by EUCLID in his *'Elements'*), has to be considered as an *art*, and not as a *science*.

It is interesting to examine whether the Aristotelian arguments are supported by the results of the more modern investigations concerning the foundations of geometry.

It was proved by FELIX KLEIN (1870/1871) [on the basis of some ideas of EUGENIO BELTRAMI (1868)] and (with full rigor) by DAVID HILBERT (in 1899 and 1903) that the Euclidean postulate of parallel lines is independent of the remaining postulates (for more details, see p. 5 & pp. 245-263) in KAROL BORSUK & WANDA SZMIELEW: *'Foundations of Geometry'*, 1960, op. cit.).

ADRIEN MARIE LEGENDRE proved, in the period between 1800 and 1833, that the validity of the *Triangle-Sum-Theorem,* which states that the sum of the three interior angles of a triangle is always equal to two right angles, is equivalent to the validity of the Euclidean postulate of parallel lines (cf. THOMAS L. HEATH: *'The thirteen Books of Euclid's Elements',* vol. 1 (1956), pp. 213-219). This proves that the *Triangle-Sum-Theorem* cannot be deduced from the *definition of the essence* of the term *'triangle'* alone! This, however, also proves that the property (*) is not necessarily a property of all triangles, and hence not a property that belongs to a triangle *per se*.

These arguments show that the validity of the *Triangle-Sum-Theorem*,
(1.) as a mathematical theorem, depends on the validity of postulates and,
(2.) as a statement concerning the real universe, depends on the truth of empirical observations (cf. ALBERT EINSTEIN (1879-1955) in his essay on *'Geometrie und Erfahrung'* ('Geometry and empirical knowledge', op. cit.).

This opens a remarkable view onto the subject discussed in this chapter:

> *"Insofern sich die Sätze der Mathematik auf die Wirklichkeit beziehen, sind sie nicht sicher; und insofern sie sicher sind, beziehen sie sich nicht auf die Wirklichkeit."*
>
> [Insofar, as the statements of Mathematics concern Reality they are not certain; and insofar they are certain, they are not concerned with Reality.]
>
> ALBERT EINSTEIN*: 'Geometrie und Erfahrung'* (1921, op. cit.

This is a highly interesting remark and, maybe, also surprising, since it contradicts the Aristotelian view. We have to accept that dependence upon empirical observations can make absolute certainty impossible.

7.5 Discussion

In the section 7.4, we presented the words of some prominent philosophers, logicians and theologians. Among those who were convinced that mathematics was not a science in the Aristotelian sense were

> ALEXANDER PÍCCOLOMÍNI, BENEDICTUS PERERIUS, SIMON SIMONIUS and ARISTOTLE himself,

and among those who wanted mathematics to be considered a science in the Aristotelian sense were

> THOMAS AQUINAS, JACOB SCHEGK, JOSEPH BLANCANUS*,* et al.

It is striking that they were all silent about the different views of ARISTOTLE and EUCLID concerning the ontological status of geometric objects. Even CHRISTOPH CLAVIUS, in his monumental edition of EUCLID*'s 'Elements'* (op. cit., Rome 1574, Cologne 1591), did not defend EUCLID*'s* position regarding the ontological status of mathematical objects. (EUCLID himself had not explicitly stated it either.) CLAVIUS only tried to defend mathematics against the accusation that it is not a science. In the preface, he writes, under the paragraph *'Nobilitas atque praestantia scientiarum Mathematicarum'* (The Fame and Excellence of the Mathematical Sciences):

> *"Quoniam disciplinae Mathematicae de rebus agunt, quae absque ulla materia sensibili considerantur, quamvis re ipsa materiae sint immersae; perspicuum est eas medium inter Metaphysicam, et naturalem scientiam obtinere locum, si subiectum earum consideremus."*
>
> [Because the mathematical disciplines treat of things which are considered detached from all sensible matter, even though they are immersed in matter, it is clear that they occupy a place between metaphysics and natural science when we look at their subject areas.]

This is, again, the Aristotelian position: mathematics deals with the sense-perceivable objects of the world. But it deals with them only with respect to number, length, breadth, depth or form, ignoring all other properties, but in such a way as if they were objects in their own right. (We discussed this in chapter 3.)

CLAVIUS thinks that mathematics therefore occupies a place between metaphysics and the natural sciences. But he goes on to assert that only in mathematics can one obtain statements of absolute certainty. In the natural sciences, the results depend on many

empirical observations, and in metaphysics, one stands indecisively between the views of the Peripatetics, Greek, Latin and Arab commentators, nominalists and realists, etc. Mathematics, then, should have first place among all the sciences.

The dispute over the status of mathematics continued. But, from the early 17th century onwards, interest in the great controversy of whether or not mathematics was a science in the Aristotelian sense gradually waned. Contributing to this was the emergence of new physical theories and new mathematical disciplines (especially algebra, also known as 'Coss', projective geometry, analytical geometry and differential and integral calculus), which obviously did not fit into the Aristotelian scheme.

With the end of the Renaissance, the Aristotelian theory of science gradually lost its supremacy and gave way to a new ideal of science, which resulted from a modification of the Euclidean axiomatic theory. The prestige that EUCLID's axiomatic method had meanwhile gained led to it also being applied in areas other than mathematics. It was then said that these areas were dealt with *'more geometrico'*, i.e.. *in the manner of geometry.* A famous example of this is the *'Ethics'* of the Portuguese-Dutch philosopher BENEDICTUS DE SPINOZA (1632-1677).

But the big issue, however, had not been decided. It therefore continued to occupy the mathematicians, logicians and philosophers in the following centuries.

(i) Is it possible to define the basic concepts of geometry and arithmetic in such a way that all valid propositions of geometry and all valid propositions of arithmetic can be proved from what is laid down in the definitions alone? The proofs would then not have to be based on postulates. The valid propositions of these disciplines would yield knowledge a priori and, in the sense of KANT, *analytical judgments* (cf. Chapter 12). They would be *apodictically certain* and supported by *irrefutable, unconditional proofs.*

(ii) Is it possible to realize the program specified in (i) with *'definitiones essentiales'?* - If the answer is negative, the question arises as to whether the program can be executed with the more meaningful *causal definitions* (also called *'genetic definitions'*)? We return to this question in Chapter 10.

(iii) But the question also arises as to whether the basic concepts of geometry and arithmetic are explicitely definable at all. - We will discuss this in Chapters 10 and 20.

It may be quite surprising, however, to note that the question posed in (i) has a positive answer when the term *'definition'* is not understood in the Aristotelian sense as a *'definition of essence'* but in the sense of modern structuralist mathematics as an *'implicit definition'*. In fact, in modern structuralist mathematics the axioms of a mathematical theory are nothing else than *'implicit definitions'* of the basic concepts of that theory. These implicit definitions are, hence, the only sources from which we draw when we prove mathematical theorems. This, in fact, is a quite convincing answer to the old question posed above in (i). (See Chapters 19 and 20 for more details.)

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Chapter 8
The Cartesian Nativism

The Prometheus Myth, Augustinian Illuminism and Cartesian Rationalism

In ancient times, epic poets repeatedly invoked the Muses to inspire them and guide them in the writing of their songs. The Muses are the daughters of Zeus and Mnêmosynê; they are the goddesses of song, knowledge and memory. The invocation of the Muses is based on the conviction that they, as goddesses, attend all events, and therefore have proper knowledge of everything.

Moreover, the language of the poets was understood as the language of the Gods, because it is shaped by rhythm (and, in the Occident, also by rhymes and alliterations) and, in its dignity and elevated style, is not the ordinary language of humanity.

In the sanctuary of Delphi, Pythia, after having drunk water from the inspiring holy well, and thus *'being full of the god',* and after having put herself into a state of intoxication by chewing laurel leaves, expressed in hexameters what the God Apollo (Apollon Pythios) had communicated to her.

In his dialogue *'Ion'*, PLATO lets SOCRATES say that the poets are also the spokesmen of the Gods, and that their beautiful poems are not human, but *'divine from the Gods'* (534e), and that it is the Gods themselves who speak to us through the mouths of the poets (534d).

8.1 The Divine Origin of Mathematics

In a mathematical work, one searches in vain for the invocation of the Gods or the Muses. This is probably because there is no Muse for mathematics among the nine Muses. There is a Muse of comedy and idyllic poetry (*Thália*), a Muse of tragedy (*Melpoménê*), a Muse of epic poetry (*Kalliópê*), etc., and even a Muse of astronomy (*Uranía*). But, nevertheless, many mathematicians believe that mathematics is also of divine origin. For example, RENÉ DESCARTES (1596-1650) wrote, in a letter to MARIN MERSENNE (1588-1648) dated April 15, 1630:

> *"Les verités mathématiques, lesquelles vous nommé eternelles, ont esté establies de Dieu."*
> [The mathematical truths that you have called eternal truths have been established by God.]
> R. DESCARTES, cf. Œuvres de DESCARTES, volume 1, 1974, p. 145.

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Science Networks. Historical Studies 62, https://doi.org/10.1007/978-3-031-27304-9_8

Similar remarks can be found in the writings of many other mathematicians, from the early days of mathematics to the present day. The following statement is handed down from SRINIWASA RAMANUJAN (1887-1920):

"An equation has no meaning to me unless it expresses a thought of God."

Such convictions are scattered all over the world in the most diverse cultures. In Europe, we can pursue the traces of such beliefs back to the early days of Greek culture. They are already evident in the Prometheus myth (HESIODUS, AESCHYLUS, PLATO), and then again in the Stoa, Neoplatonism, Patristics and the Renaissance.

In HESIODUS (Ἡσίοδος: *'Theogony',* verses 507-616, and *'Works and Days',* verses 45-89), Prometheus is one of the Titans, one who brought fire down from heaven to the humans and taught them the use of it. In PLATO (*'Philebos'*, 16c-d, and *'Protagoras'*, 320c-322a), the figure of Prometheus is developed further into a God who brought humans not only fire, but also many arts and the light of knowledge. In AESCHYLUS (*'The Bound Prometheus'*), it is Prometheus who gave humankind the art of medicine, calculating with numbers, writing with letters, etc. AESCHYLUS lets Prometheus end the long list of all of the benefits he has brought to humankind as follows:

'In a word, all human art and all science is my gift.'

Prometheus therefore not only brought fire to humans, but also the *'light of knowledge'*. He gave the human race the sciences, and especially mathematics.

The Prometheus myth is only a poetic formulation of the conviction that all arts and sciences are of divine origin. In the Stoá, this conviction was even given a very central place.

8.2 The Greek and Roman Stoics

One of the four great schools of philosophy in Athens[1] was the 'Stoá' (Στοά), founded by ZENO (Ζήνων) of Kition around 310 B.C.E. It is called 'Stoa' after the place of assembly, the *'Stoa Poikilê'* (στοὰ ποικίλη), in the centre of Athens. This 'Stoa Poikilê' was an open foyer decorated with large colorful mural paintings, with a roof supported by high columns.

The Stoics taught that the world was arranged by the deity to its best,[2] and that everything was planned in advance by it. In particular, the deity had given man the gift of intellect. The Stoic LUCIUS ANNAEUS SENECA (ca. 4 BCE - 65 CE) expressed this thought (in his 66th letter to Lucilius) as follows:

"Ratio autem nihil aliud est, quam in corpus humanum pars divini spiritus mersa."
[But the mind is nothing but a part of the divine spirit sunk into the human body.]

According to the Stoics, the divine mind is scattered like a seed throughout the universe and is active in all men. The whole human race participates in the *logos spermatikos* (λόγος σπερματικός). The Stoic EPIKTÊT (Ἐπίκτητος, ca. 50-138 C.E.) was convinced,

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[1] These were PLATO*'s Academy* (cf. Chap. 2), ARISTOTLE*'s Perípatos* (cf. Chap. 3), EPICURUS*' Garden* and ZENO*'s Stoa*. EPICURUS lived from 341 to 270 B.C.E.

[2] Which, as is well known, VOLTAIRE denied, and which he smothered in derision in his *'Candide'*.

'that we share our body with the animals, but that we share our intellect and our reason with the Gods' (cf. ARRIAN: *'Unterredungen mit Epiktet*', I,3, Diederichs Verlag Leipzig 1905, p. 9).

One can see from these quotations that the old Prometheus myth still had a great effect on the philosophy of the Stoics and that it raised mathematics, along with a number of other disciplines, to the rank of a field of activity of the Gods.

The emerging philosophy of Christianity, from the third century onwards, absorbed the Platonic doctrine of ideas, the New Platonism and also the philosophy of the Stoa. The Platonic *'ideas'* thereby became *'thoughts of God'*. In particular, the objects of mathematics became *'thoughts of a deity'* in this transformation.

8.3 The mathematical objects as thoughts of God (AUGUSTINUS)

AURELIUS AUGUSTINUS (ST. AUGUSTINE, 354-430) is one of the most prominent representatives among the patristic writers. He contributed to the doctrines of the Platonists, Stoics and New Platonists (PLATO, SENECA, PLOTINOS, PORPHYRIOS and others) being reinterpreted in the sense of Christianity. He taught that all of our cognitions are nothing more than *'inner revelations of God'*. This is apparently a reinterpretation of the Platonic doctrine of recollection (anamnesis), about which we have reported in Chapter 2, but, of course, it is also a reference to the myth of the virtue of the Gods and the Muses (as was explained above).

In his dialogue *'De Magistro'* (written around 390), AUGUSTINUS deals with the function of language. In his opinion, statements that we read or hear cause ideas to rise up in us. If the statements are about things of the natural real world, then the ideas are memories of perceptions we have previously formed. If the statements are about objects that are only spiritually perceptible, then the ideas that arise in us are *'inner revelations'* of a deity. AUGUSTINUS writes about his own role as a teacher:

"... docetur enim non verbis meis, sed ipsis rebus deo intus pandente manifestis."

[... (the disciple) is not taught by my words, but by the things themselves, which become tangible (=visible) through God's inner revelation.]

AUGUSTINUS: *'De Magistro'*, § 40, op. cit. p. 370.

In order to achieve cognition and to come to the knowledge of the truth, our rational soul questions the wisdom of God (*'De Magistro'*, § 38). All cognition, including the cognition of mathematical truths, is, for AUGUSTINUS, an inner revelation of God (*'De Magistro'*, § 40). The truth can be recognized only if a higher being shows it to us (AUGUSTINUS: *'Contra Academicos'*, § 5). We catch sight of the truth in the (supernatural) light of God. The human spirit is *illuminated* by the divine spirit. The doctrine of AUGUSTINUS, which is indicated here, is therefore called *'Illuminism'* (cf. ALEXANDER KOYRE, op. cit., p. 166 ff.).

The Platonic doctrine of ideas was not rejected by AUGUSTINUS. He merely determined the place of the world of ideas in a new way and transferred it to divine reason. In his writing *'De diversis quaestionibus octoginta tribus',* quaestio 46: *'De ideis'*, we read, for example: *"ideae ... quae in divina intelligentia continentur"*. From this, it follows that

mathematical objects also have their place in the divine intellect.

This statement contains an assertion about the ontological status of mathematical objects, but nothing about their epistemic status. It merely states that what we know is a gift from God, that is, merely an outflow of divine knowledge. JOSÉ ORTEGA Y GASSET (1883-1955), in his book *'En Torno a Galileo'* (1933), very succinctly summarized AUGUSTINE's view as follows:

> *"El hombre por si solo no es capaz de pensar la simple verdad: 2+2=4. La intuitión de toda verdad, eso que llamamos sensu stricto inteleccion, es operación de Dios en nosotros."*
>
> [Man is not able to think the simple truth: 2+2=4 by himself. The intuition of all truth, for which »insight« can also be said in the exact sense of the word, is an operation of the Gods in us.]
>
> J. ORTEGA Y GASSET: *'En Torno a Galileo, esquema de las crisis',* p. 183, op. cit.

ARISTOTLE, AUGUSTINUS, PLATO, PLOTINUS and others answered the question of the 'where from' of that which we know in very different ways. But what is known is, for none of them, a creative contribution of the human spirit on its own authority, but only something that exists either in the realm of the thoughts of a God or in the realm of that which exists by itself, in the world of ideas or in the sensorially perceivable real world as a fact. That, for example, *'three times three equals nine'* is, according to this view, true because it is true in both the world of thoughts of a God and also in the area of the intrinsically existent. The creation of the concept of the number and the establishment of the rules of calculating with numbers are not regarded here as creative achievements of humans, and it does not matter whether humans take note of the truths of number theory or not. They exist independently of them:

> *'That three times three is nine, ... is necessarily true even then when all mankind snores.'*
>
> AUGUSTINUS: *'Against the academics'*, op. cit., p. 122.

8.4 RENÉ DESCARTES: Mathematical laws as edicts of a deity

The question as to why (for example) $2+2=4$ holds true was probably never asked by AUGUSTINUS. In fact, it is probably true that nobody asked themselves this question for quite a few subsequent centuries. They were satisfied with the answer that a God had set it up that way.

Even RENÉ DESCARTES (Renatus Cartesius, 1596-1650) was still satisfied with this answer about 1200 years later. In his youth, DESCARTES had been pupil (alumnus) of the Jesuit College 'Collège Royal' in La Flêche (about 65 *km* north-west of Tours), and had been thoroughly schooled in the philosophical and theological doctrines of AUGUSTINUS and THOMAS AQUINAS there, as well as in the doctrines of the Greek and Roman Stoics. DESCARTES thus became an heir to the philosophy of the Stoics and, in particular, also to Augustinian philosophy.

DESCARTES compared God to a sovereign legislator who established both the physical laws and the mathematical laws according to his will,

"ainsy qu'un Roy establist des lois en son Royausme".
[so to speak, like a king decrees the laws of his kingdom.]
DESCARTES, Œuvres (J. Vrin, Paris, 1974), volume 1, p. 145).

In a letter to MERSENNE dated May 27, 1630, he wrote that God could have arranged it too,

"... qu'il ne fust pas vray que toutes les lignes tirées du centre à la circonference fussent égales". [... that the radii of a circle are not all the same.]
DESCARTES, Œuvres ,J. Vrin, Paris, 1974, volume 1, p. 152.

This, however, does not make sense, because circles are, by definition, such closed plane curves in which all *'straight lines drawn from a certain point within the figure to the circumference are equal to one another'* (EUCLID: *'Elements'*, Book I, Definition 15). - DESCARTES wrote, in the years 1636-1640, in his *'Meditationes*':

",... nec voluit tres angulos trianguli aequales esse duobus rectis."
[...and as much as he (God) wanted the angles of a triangle to be two right angles in total.]
R. DESCARTES: *'Meditationes de prima Philosophia, Sextae Responsiones',* Œuvres (J. Vrin, Paris, 1973), volume 7, page 432.

In non-Euclidean geometries, the interior angles of a triangle do not add up to two right angles. The laws of such a geometry could have actually been enacted by the 'sovereign legislator' of our natural world, and perhaps he even did so, as ALBERT EINSTEIN conjectured as a possibility.

But DESCARTES was equally convinced that the arithmetic equations $2 \cdot 4 = 8$ and $2 + 3 = 5$ (op. cit. Volume VII, p. 436, p. 445; Volume IX-1, p. 28) also originated from the divine will. He wrote about it:

"Nec proinde putandum est aeternas veritates pendere ab humano intellectu, vel ab aliis rebus existentibus, sed a solo Deo, qui ipsas ab aeterno, ut summus legislator, instituit."
[And therefore one must not believe that the eternal truths depend on the human mind, or on the existence of any things, but solely on the divine will, which, like a sovereign law-maker has established them for all eternity.]
R. DESCARTES: *'Meditationes de prima Philosophia, Sextae Responsiones*' 'Œuvres (J. Vrin, Paris, 1973), Volume 7, page 436.

Since mathematical facts depend on divine will, an atheist (according to DESCARTES) cannot really know them:

"Quantum ad scientiam Athei, facile est demonstrare illam non esse immutabilem & certam."
[As for the science of an atheist, it is easy to show that he cannot know anything with immutability and certainty.]
R. DESCARTES: *'Meditationes de prima Philosophia, Sextae Responsiones',* Œuvres (J. Vrin, Paris, 1973), Volume 7, page 428.

But could a God have issued $2 + 3 = 6$ or any other equation instead of $2 + 3 = 5$? He could not have done so, because the identity 2+3 = 5 follows (using the associativity law) directly from the definitions 2 = 1+1, 3 = 2+1, 4 = 3+1 and 5 = 4+1 as follows:

$$5 = 4 + 1 = (3 + 1) + 1 = ((2 + 1) + 1) + 1 = (((1 + 1) + 1) + 1) + 1 = 2 + ((1 + 1) + 1) = 2 + 3.$$

This was pointed out by GOTTFRIED WILHELM LEIBNIZ (1646-1716) in his dialogue *'Confessio Philosophi'* (op. cit., pp. 50-51). (LEIBNIZ, however, had discussed the Augustinian example $3 \cdot 3 = 3^2 = 9$ there [3]).

Already in antiquity, CAIUS PLINIUS SECUNDUS (he was born in the year 23 or 24 and died in 79) had remarked, in his *'Natural History'* (2nd book, § 11), *'that it would be a mistake to assume that the Gods....were omnipotent, and that they could not cause that for example* $2 \cdot 10 \neq 20$ *would be true.'*

DESCARTES himself had also written, in his *'Regulae ad Directionem Ingenii'* (1628), that, for example, 4+3 = 7 is 'necessarily true' ("*haec compositio necessaria est"*, cf. Œuvres X, p. 421) and that we could not think otherwise. DESCARTES seems to contradict himself. But he does not do so, for, in a letter to MERSENNE (of May 27, 1630, cf. Œuvres I, p. 152), he adds that, in the world in which we live, for example, 4+3 = 7 is a necessary truth, but that a God, in his omnipotence and freedom of will, could have created a completely different world in which the objects would obey a different arithmetic and a different geometry. In this respect (according to DESCARTES' opinion), arithmetic and geometrical truths can depend on the divine will. So, there is no contradiction within the statements of DESCARTES, LEIBNIZ and PLINY.

Such considerations, however, do not contribute to the foundation of mathematics, but they do show how much DESCARTES was still rooted in the thinking of the patristic and scholastic philosophy.

8.5 DESCARTES' nativism

The question of how we humans can succeed in recognizing the 'eternal truths' of mathematics was answered differently by DESCARTES than by PLATO and AUGUSTINUS. He did not refer to the *'Nous'* and the *'recollection of prenatal knowledge'*, as PLATO taught it, nor to the *'inward divine revelation'* of which AUGUSTINUS spoke, but to a *'direct contemplation of the ideas that are innate in us'*. DESCARTES, in leaning upon CICERO and THOMAS AQUINAS, spoke of 'innate ideas' (*ideae innatae*).[4] The view that certain ideas,

[3] See also LEIBNIZ*'s 'Neue Abhandlungen über den menschlichen Verstand',* F. Meiner Verlag Hamburg 1971, p. 490.

[4] The Latin term *'idea innata'* was introduced by CICERO in his work *'De Natura Deorum',* I,43-45 (op. cit. pp. 44-46), and later also used by THOMAS AQUINAS (in his commentary on the Trinity-Treatise of BOETHIUS, Quaestio III) and others. DESCARTES wrote, in a letter of April 23, 1649, to CLERSELIER, that he had adopted the doctrine of innate ideas in order to be able to contradict those who say that the concept of God had been shaped by us humans: *"pour prevenir l'opinion de ceux qui pourroient dire que l'idee de Dieu est faite par nous"* (DESCARTES, Œuvres V, p. 354). Obviously, DESCARTES is following EPICURUS here, the Greek philosopher, and also CICERO - cf. CICERO in his book *'De Natura Deorum'*, book I, Chapter XVI, #43 -45 and Chapter XVII.

In his writings. *'Regulae ad directionem ingenii'* (ca. 1628) and *'La Recherche de la verité par la Lumière naturelle'* (DESCARTES, Œuvres X), DESCARTES calls the innate ideas, following the Stoics, *"semina"*, and writes that they were made conscious *"per lumen quoddam ingenitum"*.

concepts and basic insights, etc., are innate is called *'nativism'*.

DESCARTES thought that he had many ideas of things in his mind that would not exist outside of his thinking and were not created by his mind, and therefore must have been present in his mind from birth. In his *'Meditationes',* Objectiones Quintae, he expressed himself as follows:

> *"Sed quantum ad essentias quae clare & distincte cognoscuntur, qualis est ea trianguli alteriusve cujusvis figurae Geometricae, facilè cogam te ut fatearis illarum ideas, quae in nobis sunt, a singularibus non esse desumtas."*
>
> [However, as for the entities that are recognized in a *clear and distinct* manner, such as the triangle and the other geometric figures, it will be easy to force you to admit that their *ideas,* which are within us, are not taken from individual phenomena.]
>
> DESCARTES, *Œuvres* VII (*'Meditationes'*), p. 380.

In DESCARTES' opinion, the ideas of geometric figures have thus not entered us through the senses (loc. cit., p. 381). We have them as *innate ideas* within us, and can therefore recognize the drawn, sense-perceptible straight lines, triangles, circles, etc., as lines, as triangles, as circles (in their true ideal forms), etc. He writes about this:

> *"Ac proinde, cum primum olim in infantia figuram triangularem in chartâ depictam aspeximus, non potuit illa figura nos docere quo pacto verus triangulus, ut a Geometris consideratur, esset concipiendus, quia non aliter in ea continebatur quam velut in rudi ligno Mercurius. Sed quia jam ante in nobis erat idea veri trianguli, & facilius a mente nostra, quàm magis composita figura picti trianguli, concipi poterat, idcirco, visâ istâ figurâ compositâ, non illam ipsam, sed potius verum triangulum apprehendimus. Eodem plane modo quo, dum respicimus in chartam, in quâ lineolae atramento ita ductae sunt ut faciem hominis repraesentent, non tam excitatur in nobis idea istarum lineolarum, quam hominis: quod omnino non contingeret, nisi facies humana nobis aliunde nota fuisset."*
>
> [And in the same way, when in our youth we saw for the first time the representation of a triangular figure on a piece of paper, that figure could not teach us how a real triangle, as seen by geometers, should be understood, because it was contained in the figure in no other way as e.g. the figure of Mercurius in a rough woodcut. But because the idea of a true triangle was already present in us before that, and because it was easier for our mind to grasp it than the more composite figure of the drawn triangle, we did not grasp it itself when we saw that composite figure, but rather the true triangle. Similarly, when we look at a sheet of paper on which black lines have been drawn to represent the face of a person, these lines do not evoke the idea of those lines, but the idea of the person. And this would not be possible at all if the human face had not been known to us from another side.]
>
> DESCARTES: *'Meditiones de Prima Philosophia'*, Œuvres VII, p. 382.

DESCARTES apparently believed that the ideas of geometric points, straight lines, circles, triangles, etc., are innate to us. He also believed that we are always able to tell whether any simple scribbling presented falls under any of the above innate concepts. If, for example, we see a scribble consisting of three connected, reasonably straight lines, then, according to DESCARTES' view, we are able to recognize that the scribble refers to a triangle. The *'idea of a triangle'* is simpler (and occurred earlier in our mind) than the concretely scribbled triangle, and thus is easier to recognize in its form.

8.6 The *ideas* of mathematical objects

According to DESCARTES, we are familiar with the *'ideas'* of the objects of arithmetic and geometry because they are innate to us. But what are *'ideas'* and how do we recognize the *'innate ideas'*? DESCARTES says very little about this.

For PLATO, 'ideas' are *common images of appearance* of *all things* that belong to a certain genus or species (cf. Chapter 2, section 2.5). For DESCARTES, however, only those contents of our consciousness (or of our world of thoughts) should be called *'ideas'*, which are appearances of *single things* (cf. his 3rd meditation, Œuvres, vol. VII, p. 37; vol. IX-1, p. 29).

DESCARTES believes that all ideas that are *'clear and distinct'* in our mind (or our consciousness) are the ideas of truly existing things: *'vera entia'*. To prove this, he says that such ideas stand for objects of which we can, in principle, know everything - precisely because they are *'clearly and distinctly'* perceptible. It is only in regard to truly existing objects that we can (at least in principle) know everything. As regards fictions, i.e., objects that we have invented ourselves, we can only know that which we ourselves have put into their descriptions. We cannot (in DESCARTES' opinion) give descriptions of fictitious objects in which it is determined whether each property is true for that object or not. It follows from these considerations that the objects of arithmetic and geometry, whose ideas all appear *'clear and distinct'* in our minds, are not fictions, but truly existing things.[5] According to DESCARTES, they are *'vera entia'* (DESCARTES, Œuvres V, p. 160; cf. also G. BROWN, R.G.KOTTICH and Å. PETZALL, op. cit.).

What is problematic here is the conclusion, from the *'clear and distinct'* perceptibility of the appearances of the ideas in our minds, of the real existence of what is only depicted. Do we mean an existence that is outside of the world of thought, independent of one's own thinking? Where is the place of existence of the things the ideas of which manifest in our consciousness? DESCARTES says nothing about this.

DESCARTES uses the adjectives *'clear and distinct'* so often that they have become key words in his philosophy. For him, these and the following adjectives and adverbs are all synonymous:

"claire & distinct" [*clear and distinct*], *cf.* Œuvres VI, p. 18),
"*tres-clairement & tres-évidemment"* [*very clear and completely obvious*]
cf. Œuvres IX,1, p. 51),
"claire & assurée" [*clear and certain*], *cf.* Œuvres VI, p. 4) or
"precis & exact" [*precise and sharp*], *cf.* Œuvres VI, p. 389), etc.

The terminology here is of Greek origin. ZENO of Kition, the stoic philosopher, introduced the term *'enargeia'* (ἐνάργεια, i.e., *'clarity', 'distinctness'*) to denote that which is self-evident. '*Enargês'* (ἐναργής) means *'clear', 'distinct'*. CICERO proposed the

[5] A thought that is very closely related to DESCARTES's thought was expressed by KURT GÖDEL in his *'*GIBBS-*lecture'* of 1951 (cf. GÖDEL*'s 'Collected Works'*, Volume III (Oxford 1995), p. 311). We will return to this in Part III in the discussion of modern Platonism. It is, however, a thought that goes back to the stoic philosopher ZENO of Kition and even to PARMENIDES.

following Latin translations[6]: *'perspicuus et evidens'*. Here, *'perspicuus'* means *'transparent', 'clear', 'lucid',...* and *'evidens'* means *'evident', 'visible', 'obvious'*. It seems to us that DESCARTES has borrowed the terms *'claire & distinct'* from ZENO, resp. from CICERO. This is good to know, since it gives a hint as to the meaning that is intended.

For DESCARTES, as for ZENO and CICERO, the *'clear and distinct'* perception of an idea should give us certainty that it is an idea of a truly existing thing. Such an idea had been called *'kataleptike phantasia'* (καταληπτικὴ φαντασία) by ZENO of Kition. Here, *'intuition'* is a good translation into English of the Greek term *'phantasia'*. Thus, an intuition is *'kataleptical'* if it is comprehensive, i.e., if the object intuited is *grasped with all its properties*. Intuition in general does not necessarily involve the real existence of the object intuited. But if the intuition is *'clear and distinct',* that is, if it is *'kataleptical'*, then, according to ZENO, the object intuited really exists. From SEXTUS EMPIRICUS[7] (in his treatise *'Against the Logicians'*, book 1, §152), we read:

> "Καταληπτικὴ δὲ φαντασία κατὰ τούτους ἐτύγχανεν ἡ ἀληθὴς καὶ τοιαύτη οἵα οὐκ ἂν γένοιτο ψευδής.
>
> [A comprehensive (i.e. kataleptical) intuition, (according to the Stoics), is an intuition which is true and of such a kind as to be incapable of becoming false.]

Thus, according to the Stoics, if the perception of an idea is *'clear and distinct'*, then the idea is kataleptical and refers to a truly existing object. As we have seen above, it was also a firm belief of DESCARTES that ideas that are *'clear and distinct'* in our minds are not fictions, but truly existing objects: *'vera entia'*.

However, it is perhaps surprising that, for DESCARTES, certainty in mathematics is not obtained through careful, step-by-step *deductions* from precisely stated definitions and axioms or postulates, but through *'clear and distinct'* perceptions of ideas to which DESCARTES gives the name of *'intuition'*.

8.7 DESCARTES' concept of *'intuition'*

Our intellect succeeds in *'directly contemplating the ideas innate in us'* through an activity that DESCARTES calls *'intuition'*. In his essay *Rules for Direction of the Mind* (*'Regulae ad Directionem Ingenii',* written ca. 1628, Opuscula Posthuma, published in Amsterdam 1701, Œuvres X), he claims (in the commentary on the third rule):

> *"Sed ne deinceps in eumdem errorem delabamur, hic recensentur omnes intellectus nostri actiones, per quas ad rerum cognitionem absque ullo deceptionis metu possimus pervenire: admittunturque tantum duae, intuitus scilicet & deductio."*
>
> [But lest we fall into the same error, here all the activities of our intellect through which we can arrive at the knowledge of things without any fear of deception are to be examined: only two are permissible, namely *intuition* and *deduction*].
>
> R. DESCARTES: *'Regulae ad Directionem Ingenii',* Œuvres (J. Vrin, Paris, 1974), Volume 10, page 368.

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[6] cf. CICERO*: 'Academica'*, second book in its first draft, called "Lucullus", VI,17, op. cit., p. 488-489.

[7] Among the 11 books *'Adversus mathematicos'*, books 7 - 11 are titled *'Adversus dogmaticos'* (Πρὸς δογματικούς). And among these, books 7 and 8 are titled *'Against the Logicians'* (Πρὸς λογικούς).

DESCARTES then explains the term 'intuition' as follows:

> *"Per* intuitum *intelligo, non fluctuantem sensuum fidem, vel malè componentis imaginationis judicium fallax; sed mentis purae & attentae tam facilem distinctumque conceptum, ut de eo, quod intelligimus, nulla prorsus dubitatio relinquatur; seu, quod idem est, mentis purae & attentae non dubium conceptum, qui à solâ rationis luce nascitur, & ipsâmet deductione certior est, quia simplicior, quam tamen etiam ab homine malè fieri non posse suprà notavimus. Ita unusquisque animo potest intueri, se existere, se cogitare, triangulum terminari tribus lineis tantùm, globum unicâ superficie, & et similia."*
>
> [By *intuition* I do not mean the unstable testimony of the senses or the deceptive judgment based on the confused images of sense-perception, but the immediate and discriminating comprehension of the pure and attentive mind, so that there is no doubt at all about what has been recognized, or, which is the same thing, the comprehension of the pure and attentive mind, which is beyond all doubt, and which comes from the light of reason alone. *Intuition* itself is even more certain, because it is easier than *deduction*, which, as has been noted above, can be carried out by us without error. Thus, through intuition, everyone can grasp with his mind that he exists, that he has consciousness, that the triangle is bounded by three sides, the sphere by a single surface, and so on (.....).]
>
> R. DESCARTES: *'Regulae ad Directionem Ingenii',* Œuvres (J. Vrin, Paris, 1974), Volume X, page 368.

Whether intuition is *'more certain than deduction'* can be doubted. For DESCARTES, at any rate, intuition is the source and basis of all knowledge, even in the field of mathematics.

The (Middle Latin) word *'intuitio'* generally refers to immediate mental perception, the *visio intellectualis*, the non-discursive, non-reflective recording of facts or processes. The word is derived from *'intueri'* ("to look at something very carefully, to observe, to consider").

For DESCARTES, 'intuition' is the *'immediate and distinctive comprehension of the pure and attentive mind'*. That what is intuited are the ideas. In the case of the innate idea of a triangle, for example, it is possible (in DESCARTES' opinion) for the human mind to look at this idea and to recognize it *'clearly and distinctly'* and to 'see' immediately that a triangle has three sides, that the interior angles add up to two right angles - and, as we might add, somewhat sceptically, that, consequently, the Euclidean parallel postulate is true, etc. (cf. DESCARTES VII, p. 432; IX-1, p. 233). The mind can gain these insights directly through intuition, i.e., through an attentive look at the ideas that appear to us in our minds.

From this discussion, we can see how, in DESCARTES' view, mathematical insights can be gained. Mathematical theorems can be obtained either through *deduction* from already known theorems or through *intuition*, i.e., through a *'clear and distinct'* grasp of the ideas underlying the objects under consideration.

DESCARTES was of the opinion that he could therefore draw the geometrical and arithmetical truths entirely *from within himself*, from the *ratio* (understanding, reason) alone, i.e., without having to refer to sense-perception or unproven postulates. It became familiar to call such a conviction *rationalism.*

Following these patterns, DESCARTES constructed *Analytic Geometry* in his essay *'La*

Géométrie'. The essay appeared as an appendix to his *'Discours de la Méthode'* (Leiden, 1637).

8.8 DESCARTES' Essay *'La Géometrie'*

This essay is a small book whose importance for the further development of mathematics can hardly be overestimated. In the essay, a new mathematical discipline is created, so-called *'Analytical Geometry',* on whose fundament the *differential- and integral-calculus* (GOTTFRIED WILHELM LEIBNIZ 1675/1684) and the calculus of fluxions (ISAAC NEWTON 1672, 1687, 1704, 1736) could be constructed about 40 years later.

The presentation is free of philosophical arguments. But DESCARTES' basic philosophical convictions are nevertheless clearly recognizable, since he wants to extend classical Euclidean geometry (see Chapter 4) only through such objects whose underlying ideas can be perceived by our mind *'precisely and sharply'* (*"precis & exact", cf.* Œuvres VII, p. 389). Such objects are '*vera entia*' and, in this respect, the geometric calculus deals only with objects *'which truely exist'* and delivers only *true statements.*

Euclidean geometry is a geometry that only deals with points, straight lines and circles. DESCARTES wants to extend Euclidean geometry to such curved lines that *'allow an exact and sharp measure'* (*"qui tombent sous quelque mesure precise et exact"*, Œuvres VI, p. 392). He shows that these are curves that can be drawn with instruments, in which all parts can slide frictionless against each other or along straight lines. Such instruments are related to ordinary compasses. The curves drawn with them have *'clear and exact'* descriptions and are called *'geometric curves'* by DESCARTES. Examples of such curves are the *circles, ellipses, parabolas* and *hyperbolas*, the *conchoid* of *Nicomedes*, the *kissoid* of DIOCLES (Œuvres de DESCARTES, vol. VI, p. 390), the *Cartesian leaf* (folium cartesii) and many others.

The *parabolas* are the locus lines of thrown bodies, the *ellipses* are (according to JOHANNES KEPLER) the locus lines of the planets revolving around the sun.

Curves that can only be drawn with mechanical aids, whereby physical properties (such as gravity, adhesion, friction, uniform speed, etc.) must be taken into account, are not permitted in Cartesian geometry. Such curves are called *'mechanical curves'*. Examples of such curves are the *chain line ('catena',* the line of a sagging heavy chain), the *Quadratrix* of HIPPIAS of Elis, the *Archimedean spiral*, the *cycloid* (also called a *'roulette'*) of GALILEO (1599), the trigonometric curves *'sine', 'cosine'*, '*tangent'*, etc. (Œuvres, VI, p. 390).

The criterion for the acceptance of a curve in Cartesian geometry is obviously that it is possible to get a *'clear, precise and exact'* picture of its complete course in the mind. For a curve to be *'clearly and distinctly'* graspable by our mind, it is necessary that the points lying on it

> 'are in a precise measurable relationship to each other' (*"qu'on puisse mesurer exactement",* DESCARTES, 'Œuvres', Volume VI, p. 390),

and this means that their relationships can be expressed in the language of algebra. This is what he says at the end of the discussion:

> *"(...) je ne sçache rien de meilleur que de dire que tous les poins de celles qu'on peut*

> *nommer Geometriques, c'est a dire qui tombent sous quelque mesure precise & exacte, ont necessairement quelque rapport a tous les poins d'une ligne droite, qui peut estre exprimé par quelque equation, en tous par une mesme."*
>
> [(...) I know of no better way of expressing it than to say that all the points of those curves that can be called Geometrical, that is to say that fall under some precise and exact measurement, have necessarily some relation to all the points of a straight line, which can be expressed by an equation, in a uniform way for all points].
>
> DESCARTES, 'Oeuvres', Volume VI, p. 392.

A *'geometric curve'* is thus (in today's terminology) the geometric locus of all points (x,y) that satisfy an equation (of degree k in y)

$$f(x,y) = a_0(x) + a_1(x)\cdot y + a_2(x)\cdot y^2 + \ldots + a_k(x)\cdot y^k = 0$$

with rational functions $a_i(x)$ of a real variable x (as coefficients), where each $a_i(x)$ is, hence, a quotient of two relatively prime polynomials over the field $\mathbb{R}$ of real numbers. LEIBNIZ [8] later called such curves *'algebraic curves'*.

The curves accepted in Cartesian analytical geometry are, hence, the 'geometric curves', i.e., the 'algebraic curves'. The 'mechanical curves' are also called 'transcendental', and are not accepted. (For a detailed discussion of DESCARTES' distinction between geometric and mechanical curves, see HENK BOS 1981, 1990, S. KRÄMER 1989 and PAOLO MANCOSU, op. cit., pp. 71-79).

DESCARTES explains how the description of the accepted geometric curves can be achieved through algebraic equations. He starts with the introduction of orthogonal coordinate axes (more precisely: by inserting only a right angle so that the figure under consideration lies completely between the two legs). Using some elementary theorems (e.g., the theorems III.31 [theorem of THALES] and VI.2 [*'theorem of intersection', 'Strahlensatz'* in German]) from the *'Elements'* of EUCLID, DESCARTES defines the algebraic operations of addition, subtraction, multiplication, division and extracting square roots between finite straight lines, and proves that the familiar algebraic laws are valid. With the help of these concepts, algebraic descriptions of all *'geometric curves'* are possible. It follows that these curves do have *'clear and distinct'* descriptions and, hence, are '*vera entia*'.

DESCARTES does not prove the converse of that statement, namely, that every 'algebraic curve' can also be drawn with a ruler and generalized compasses (as described above). Such a proof was only given about 240 years later, by A.B. KEMPE, op. cit.

Summary. The objects of Cartesian analytical geometry are the *really existing* geometric objects. These are all of those objects whose underlying ideas can be perceived *'clearly and distinctly'*. They are, in the sense of DESCARTES, '*vera entia*' and, in this respect, the geometric calculus deals only with objects *'which truely exist'* and delivers only *true statements*.

---

[8] cf. LEIBNIZ: *'De geometria recondita et analysi indivisibilium atque infinitorum'*, Acta eruditorum, June 1686, pp. 292-300.

In contrast to Euclidean geometry, Cartesian analytic geometry does not require a system of postulates (or axioms). There is no need to demand the acceptance of any principles, for all of the sources from which we draw when we want to prove geometric propositions are, as DESCARTES says, inside of us as *innate ideas.*

Since the human mind - according to DESCARTES′ conviction - finds the appearances of all *'truely existing'* geometric objects within itself, it can recognize what it wants to know about these objects by attentively examining their appearances. Thus, the human mind succeeds in recognizing geometric facts through *intuition* on the one hand and through *deduction* from what has already been recognized on the other. In doing so, the rational mind can use all algebraic curves, in addition to the well-known objects of Euclidean geometry. With this much richer supply of geometrical objects, it is also possible to construct and precisely describe facts that cannot be represented in Euclidean geometry.

In DESCARTES′ analytic geometry the validity of all of the postulates of Euclidean geometry can be perceived. But the antique *'methods of directed insertion'* (νεῦσις, *inclinatio*) can also be realized here. For example, the methods of angular trisection (ARCHIMEDES, NIKOMEDES, AL-KASCHI, VIÈTE, DESCARTES) and cube doubling (HIPPOCRATES of Chios, ARCHYTAS of Taranto, MENAICHMOS and others), which cannot be carried out in Euclidean geometry (and were therefore frowned upon as inexact), can be described and carried out exactly in Cartesian analytical geometry. But the theory of conic sections, which was essentially worked out by APOLLONIUS of Perge (ca. 262-190 BC) and which had no place in the geometry of the Euclidean *'Elements'*, can also be treated within Cartesian analytical geometry.

We should also mention that the method of the Persian mathematician OMAR KHAYYAM (OMAR CHAJJAM, he lived around 1070) of solving cubic equations by geometrical means using conic sections can also be carried out in DESCARTES′ analytical geometry in such a way as to be free from any objections. As solutions to cubic equations (with positive real coefficients), OMAR KHAYYAM was able to give finite straight lines of suitable lengths, but not algebraic terms. These were discovered much later by three Italian mathematicians, namely, SCIPIONE DAL FERRO (around 1515), NICOLO TARTAGLIA (around 1535) and GERONIMO CARDANO (around 1539).

8.9 Discussion

According to DESCARTES, there is only one 'true' geometry, namely, the geometry of all *'truely existing'* geometric objects, i.e., all objects for which there are *'clear and distinct'* ideas. Since, by 'intellectual contemplation', also called *'intuition'* by DESCARTES, 'straight lines' can only be imagined as actual straight lines, and these can be described very easily by linear equations, the truth of all geometrical postulates (axioms) from Book I of the Euclidean *'Elements'* is obvious. In particular, the truth of the parallel postulate in the analytic geometry of DESCARTES' is an obvious fact.

But, at the same time, it becomes clear that an 'intuitive' comprehension of a fact does not always allow for a logical analysis. For example, the intuitive content of what a 'straight line' actually is cannot be defined in purely geometric terms, and this is also the real reason why EUCLID needed the parallel postulate in his geometry.

Which other axioms are true in Cartesian analytical geometry is not so easy to answer. If one is generous, one will mention here all of the axioms of HILBERT's geometry. Then, it follows that Cartesian geometry is the *n-dimensional* geometry over the field of real numbers, as DAVID HILBERT showed in his book *'Grundlagen der Geometrie'* (1899/1902) (cf. U. FELGNER, op. cit.).

Cartesian geometry is not a *synthetic geometry*, like Euclidean geometry, i.e., a geometry in which theorems are not proved by a 'synthesis' of definitions and axioms. Cartesian geometry is an *analytic geometry* in which geometric problems can be solved by solving (i.e., analyzing) algebraic equations with the help of algebraic formalisms.

For DESCARTES, philosophical-theological convictions played a major role, since it enabled him to establish (as he thought) that the objects of his geometry are also *'truely existing objects'* and that the theorems that can be proved in his geometry can claim to be *true*.

DESCARTES' doctrine is based on the concepts of *'innate ideas'* and *'intuition'*. However, with the notion of 'intuition' comes the inextricably linked subjectivity, from which mathematics should be free. If a single mathematician is convinced that they have seen something *'clearly and distinctly'*, then it is questionable whether any other mathematician can and will also see the same thing *'clearly and distinctly'*. So, if one wants to construct mathematics on the basis of *'intuitions'*, then one must give up on the *reliability* and *certainty* of mathematics. In addition, the concept of *'intuition'* itself is rather unclear. This term is *ambiguous*, *"chargé d'ambiguité"*, as HENRI BERGSON (1859-1941) once expressed it. It has no place in the foundation of mathematics.

From about 1920 on, 'Analytic Geometry', as designed by DESCARTES, has been founded on set theory and algebra. Here, the geometric curves are sets of *n-tuples* of real numbers defined by algebraic equations. Although the geometric objects have lost their original geometric nature, the algebraic curves are now directly given and no longer need to be justified as *'vera entia'*, i.e., as truly existing geometric objects. Philosophical-theological convictions, which had guided DESCARTES, can now be completely pushed aside. In this 'purified' way, DESCARTES' 'Analytical Geometry' became one of the most important components of modern mathematics.

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Chapter 9
JOHN LOCKE's thoughts on Mathematics

"Je lus des scolastiques, je fus comme eux dans les ténèbres; je lus Locke, et j'aperçus des traits de lumière."

[I read the Scholastics, and I found myself in the Dark; I read Locke, and I kept the glimmer of light.]

VOLTAIRE: *'Le Philosophe ignorant'*, Œuvres, Kehl 1785, Volume 32, p. 92.

JOHN LOCKE was born on August 29, 1632, in Wrington (Somerset/England). He studied medicine, natural sciences and philosophy at Oxford from 1652 onwards. In 1658, he became *magister artium* and, from then on, taught as a tutor at Christchurch College in Oxford. From 1675 onwards, he lived in France, preferring Montpellier and Paris, and did not return to England until 1679. But, by 1683, he had already been forced to emigrate again. He went to Holland, and was only able to return to England in 1689. He died at the age of 72 on October 28, 1704, in Oates in the county of Essex.

During his exile in Holland, LOCKE wrote his major work: *'An Essay Concerning Humane* [sic!] *Understanding',* which was published in London in 1690. An authorized translation into French (done by PIERRE COSTE) was published in Amsterdam in 1700 (op. cit.). It was only in this translation that the *'Essay'* became widely read and had its effect on the European continent.

The '*Essay*' became one of the most influential philosophical works of the Age of Enlightenment. LOCKE had a significant influence on VOLTAIRE. In his essay *'Le philosophe ignorant'* (1767), VOLTAIRE respectfully spoke of the *wise* LOCKE (*"Le sage LOCKE"*, cf. Volume 31, p. 45, Volume 32, p. 133, in the edition of VOLTAIRE*'s* works, Kehl 1785). VOLTAIRE adopted LOCKE's thesis that all human knowledge is based on sensory perception and mental reflection, and that there are no innate ideas or innate moral laws. VOLTAIRE*'s* writings did much to make LOCKE*'s* ideas known on the European continent.

The *'Lettres à une Princesse d'Allemagne',* which LEONHARD EULER (1707-1783) wrote to SOPHIE-FRIEDERIKE-CHARLOTTE, daughter of the Margrave of Brandenburg-Schwedt, at the request of the Prussian King in the years 1760-1762, and in which EULER took up some of LOCKE*'s* thoughts, had the effect that the psychologism represented by LOCKE also found its way into the field of the foundations of mathematics.

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JOHN LOCKE*'s* main concern in his *'Essay'* is the study of the limits of human understanding. Here, psychology plays a significant role. This is so because, for LOCKE, *"the mind* (is) *the most sublime faculty of the soul"*, as he wrote in the *Epistle to the Reader* (the preface) of his *'Essay'.* For him, the activities of the *mind* are thus activities of the human *soul*.

9.1 LOCKE's doctrine of *'Ideas'*

At the beginning of the first book of his *'Essay'* (Chapter 1, § 2, p.1), LOCKE describes his intentions with the words: *"This ... being my purpose, to inquire into the ... certainty and extent of human knowledge".* LOCKE uses the word "*understanding"* (l'entendement) in the title of his book. Thus, the *Essay* is about examining what insights and what knowledge the human mind is capable of and what objects the mind can deal with at all.

A central conception here is that of an *'idea'.* LOCKE says (I,1,§8) that, with the word *'idea'*, he wants to designate everything that can be an object of the mind in its thinking. *'Ideas'* are thus, in general, the contents of consciousness, i.e., that which the human mind deals with when thinking. However, the term *'thinking'* is used here very widely, *"to cover all possible cognitive activities"* (cf. RICHARD AARON, op. cit., p. 99). In the second book (II, 8, §8, p. 74), he repeats this with a few more details:

> *„Whatsoever the mind perceives in itself, or is the immediate object of perception, thought or understanding, that I call idea."*

Thus, LOCKE*'s* use of the word *'idea'* is not the traditional one, and, in particular, not the same as PLATO*'s* or DESCARTES*'* use of it (cf. Chapters 2 and 8). While, for PLATO, *'ideas'* belong to an extramundane realm of objects, for LOCKE, they are objects that belong to the human mind. However, the definition he gave *"includes far too much"* (cf. AARON, op. cit., p. 100) and is not used in that generality in the *'Essay'*. *'Ideas',* as they occur in the *'Essay',* are always *'representations'* of something (cf. e.g. II,11,§9).

The first question LOCKE asks himself is (I,1,§ 8, p.4):
"how do ideas get into the mind?"

Before giving an answer, he dissociates himself from various philosophers (AURELIUS AUGUSTINUS and others) and, in particular, from DESCARTES. He claims that there are no innate principles and no innate ideas in the mind. He cannot prove this, of course, but he polemicises quite vigorously and discusses many examples. For example, he lets COSTE write, in the French translation of his *'Essay'* (I,1,§4, §5, p. 11-13, and also IV,7,§9), that, for example, the universally-valid sentence

> *"Il est impossible qu'une chose soit et ne soit pas en même temps."*
> ["A thing cannot be and not be at the same time."]

is obviously unknown to *children and the mentally handicapped* ("*... des petits Enfans & des Idiots",* cf. also II,11,§12)*,* and is even incomprehensible to them. So, it cannot be that all principles, which have to be recognized by all people, are in their souls from birth onwards. The abstract concept of 'Being' can certainly only be understood by people after they have come into contact with some traditions of culture and philosophy.

In the 2nd book of his *'Essay'*, LOCKE discusses the question, of how ideas come into our minds. There (II,1,§2, p. 51), he expresses the often quoted conviction that *"all our knowledge is founded on ... and ultimately derives from experience, i.e. from sensation or reflection"*. CICERO and THOMAS AQUINAS had already expressed similar thoughts. In CICERO (*'De finibus Bonorum et Malorum'*, I,64), it says

> *"Quicquid porro animo cernimus, id omne oritur a sensibus",*
> [Furthermore, everything we see with our minds comes from sensual perceptions],

and in THOMAS AQUINAS (*'Questiones disputatae de veritate',* II,3.), we read:

> *"Nihil est in intellectu, quod non fuerit prius in sensu."*
> [Nothing is in the intellect that was not previously in the senses.]

However, along with ZENO of Kition and the other Stoic Philosophers, LOCKE is convinced that not all ideas originate from sense-perception. There are also ideas in the human mind that originate from mental reflection. In fact, it seems that LOCKE was strongly influenced by a short outline, attributed to PLUTARCHUS, in which the Stoic doctrine concerning *'ideas'* is presented. The outline was contained in the famous French translation of the complete works of PLUTARCHUS, edited by JACQUES AMYOT in the years 1567-1574. It seems quite plausible that LOCKE was well acquainted with this edition of the works of PLUTARCHUS and that he read the above-mentioned outline (or draft) in this translation .[1]

Since we have reported on this outline already in section 2.5 of Chapter 2, there is no need to repeat its contents here. But let us briefly indicate that LOCKE, influenced by that which he read in it, was convinced that all ideas originate from sensory perception *and* (!) from mental reflection. He also followed the Stoics in believing that, at the moment of our birth, our consciousness may be compared with an empty piece of paper (a *tabula rasa*[2]) upon which the soul registers everything of which it becomes aware over the course of its life. There are two different kinds of registration. The registrations of the first kind are caused by sense-perception and those of the second kind rest on mental reflections on that which the soul already bears within itself. Both kinds of registration are called *experiences* (see LOCKE*'s 'Essay'*, II, 1, §2, p. 51). All of these experiences are 'ideas', according to LOCKE*'s* definition of his use of the term 'idea'. *Experience* is, hence,

- either sense-perception *('sensation')*,
- or mental introspection (*'reflection'*).

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[1] For a new, revised edition, see: '*Nouvelle Edition des Œuvres de Plutarque* in the French translation by AMYOT, revue, corrigée et augmentée par ETIENNE CLAVIER', 25 volumes, Paris 1801-1805, op. cit. The above-mentioned outline can be found there in vol. 21, as Nr. 11 in book 4 of a work with the title '*Les Opinions des Philosophes*' (Placita philosophorum), pp. 198-199. But notice that it was HERMANN DIELS who, in 1879, discovered THEODORETOS AËTIUS to be the true author of the outline. The Greek original is reprinted in the collection of fragments, edited by KARLHEINZ HÜLSER (fragment 277). HÜLSER's work is listed here at the end of Chapter 2.

[2] cf. RICHARD AARON, op. cit. p. 114 & p. 121, as well as CLEMENS BAEUMKER, op. cit. In neither publication is there any hint towards the works of AËTIOS or PLUTARCHUS with respect to the use of the term of a 'tabula rasa' or 'white paper'.

The word *'sensation'* is derived from the Latin noun *'sensatio'* and means *'perception through the senses'*. The word is a *'verbalabstractum'* derived from the Latin verb *'sentire'* and expresses the abstract type of the activity indicated in the underlying verb.

The word *'reflection'* is also a *'verbalabstractum'*. It is derived from the Latin verb *'flectere'* ('to bend', 'to bend over') and has the abstract meaning of *'bending back of the mind after an act of knowledge has been accomplished'*. Thus, *'reflection'* is self-observation of the soul, i.e., an act of the soul (or the mind) in which that which has been recognized is reconsidered.

Everything that our senses perceive, the soul also becomes aware of and forms an *idea* of. These are *ideas* that arise directly from sensations. From these *ideas*, the soul can form further *ideas* through mental reflection.

9.2 Abstraction and general ideas

One of the most important activities of the mind (or the soul) is *'abstraction'* (cf. LOCKE's *'Essay'*, II, 11, § 2). The term *'abstraction'* may be confusing, since it is used in the literature with different meanings.

There is *'abstraction'* in the sense of ARISTOTLE (cf. Chapter 3, section 3.4). The word that he used was *aphairesis* (ἀφαίρεσις) and it was ANICIUS MANLIUS TORQUATUS SEVERINUS BOËTHIUS (ca. 480-524) who proposed the Latin translation *'abstractio'* for this Aristotelian term.[3] *'Abstraction'*, for ARISTOTLE, is a method to be used in the study of the objects of a scientific theory. When the objects are only to be studied from a certain point of view, abstraction should be applied with the effect that only those properties of the objects that appertain to the science are kept in mind, and all other properties are imagined as being absent (i.e., are *abstracted*). ARISTOTLE applied the method of abstraction in geometry, which, for him, should be a scientific theory in which the objects of the real world are studied from the point of view of the shapes of their extensions. Here, *'only the quantitative and the ... continuous'* are kept in mind, and all other *'sense-perceptible properties'* of the objects are imagined as being absent (cf. ARISTOTLE *'Metaphysics'* XI, 3, 1061a29-b2). Roughly speaking, *'abstraction'* in the sense of ARISTOTLE means the action of not taking into account certain properties of objects during their investigation. Here, no new (abstract) objects are created; the objects under consideration remain the same; it is only that certain of their properties are ignored during the investigation.

'Abstraction' in the sense of JOHN LOCKE is a different kind of *'abstraction'*. It is not 'deliberate inattention' when looking at concrete objects (cf. Chapter 3, section 3.4), as in the Aristotelian case. It is an explicit elimination, liberation or freeing of certain properties, so that, in the process of abstraction, new 'abstract' objects are created that do not possess these properties.

When a particular object is perceived through the senses, then there remains a reminder of it in the soul. Out of the reminders of many similar objects, the soul eliminates all *'concomitant'* properties. These are properties that are not possessed by the objects *'per*

[3] Notice that both expressions, *aphairesis* and *abstractio*, are so-called 'Verbalabstrakta', i.e., nouns in which the general type of the activity mentioned in the underlying verb is expressed in abstracto. Notice also that the verb *'abstrahere'* in Latin means 'to carry off', 'to remove', 'to take away', 'to subtract'.

se' (καθ'αὐτό), and hence are *symbebêkos* (συμβεβηκὸς), i.e., concomitant, in the Aristotelian sense. Thereby, *'particular ideas ... become general'*. And, at this point, LOCKE states with emphasis: *"This is called abstraction"*! Here, the elimination of all concomitant properties and the elimination of the particular objects is the kind of *'abstraction'* that LOCKE has in mind! Instead of *'elimination'*, LOCKE speaks of *'separation'*. Thus, the process of *'abstraction'* reduces the totality of all properties of an object to the totality of its essential properties. The elimination of the particular object makes the reduced totality 'general'. LOCKE emphasizes that, in this way, *'particular ideas'* are transformed into *'general ideas'*. - He writes:

> *"ideas, taken from particular beings, become general representatives of all of the same kind, and their names, general names, applicable to whatever exists conformable to such abstract ideas."* J. LOCKE *'Essay'*, II,11,§9, p. 93.

This process of *"making general ideas out of particular ideas"* is a process performed in the soul (or in the mind). According to LOCKE, these *'general ideas'* also become *"general representatives of all* (particular ideas received from particular objects) *of the same kind"*. This means that the soul (or the mind) is able to create representatives that are *mental objects of the same kind*. By abstraction à la LOCKE, from a given object, a new object of a different nature is deduced.

This, however, is a very problematic consequence, which, in later times, caused a lot of disagreement (cf. Chapter 13). EDMUND HUSSERL (1859-1938), for example, called LOCKE's procedure a *'psychological hypostasis of the general' ("psychologische Hypostasierung des Allgemeinen"*, as HUSSERL puts it, op. cit., p. 121), which is as problematic as the medieval *metaphysical hypostasis of the general*. LOCKE's procedure, however, is actually more than just problematic, since it leads to logical inconsistencies. We will show this in the following section.

9.3 The abstract idea of a triangle

LOCKE illustrates his theory of abstract ideas with the example of the general and abstract ideas of 'whiteness' (II,11,§9). In book IV, he also discusses *the general, abstract idea of a triangle* (IV,7,§9, pp. 162-163). Clearly, concrete triangles are either acute-angled, or rectangled, or obtuse-angled (i.e., oblique). Hence, 'general triangles', which are neither acute-angled, nor rectangled, nor obtuse-angled, do not exist in reality. This is acknowledged by LOCKE. But he thinks that the abstract idea of a general triangle is still something that the mind (or the soul) is able to create. Such a *'general triangle'* would be a mental object *'of the same kind',* and hence a triangle in which all types of triangle are unified.

This, however, is inconceivable. LOCKE claims that such a *'general triangle'* would have a psychic being. But psychic being is always also a real being *("*LOCKE *übersieht, daß psychisches Sein auch ein reales Sein ist"*, as HUSSERL wrote, op. cit., p. 133). Thus, a 'general triangle' must be either acute-angled, or rectangled, or obtuse-angled, which shows that LOCKE's *'general triangle'* is an impossible thing.

What is wrong with LOCKE's doctrine of *'abstract ideas'* is not *'abstraction'* in itself,

but the *'hypostasis of the general'* on the basis of psychology. This was only understood in the 18th century by a few philosophers. Unfortunately, LOCKE's doctrine of *'abstract ideas'* became widely accepted, even influencing the foundations of mathematics. The effect was that the soul was considered to be capable of producing mathematical objects. During the 19th century, a number of logicians, mathematicians and philosophers classified this influence as a *'pernicious intrusion'* of psychology into mathematics. They began to eliminate psychologistic tendencies. - We refer to Chapter 13 for a more detailed discussion of the work done in that period.

9.4 LOCKE*'s* comments on the concept of the number

We now want to examine how LOCKE*'s* doctrine of 'ideas' affected his view on mathematics. We will first deal with the concept of numbers (*'Essay'*, II,16, §§ 1-8).

If one sees an object somewhere, then one can form an *idea* of this particular object and, through reflection and abstraction, also the idea of the number *'one'*. If one sees one and yet another object somewhere, then one can form the *ideas* that represent these objects individually, and then, through reflection in the mind, form the idea of the number *'two'* from both ideas. In a similar way, one obtains the ideas of the numbers *'three', 'four'*, etc. Then, *"the soul* can *go further and form ideas of larger numbers (...) without ever having seen so many things together"*, says EULER in the 100th letter to the German princess (op. cit.). According to LOCKE (II, 21, § 73), the ideas of the individual numbers are thus obtained, and they are all mental objects, created by the soul.

The ideas of natural numbers are given names (from a suitable denotation system, e.g., the decimal system as usual), in order to be able to talk about them. *'Counting'* is then nothing other than pronouncing or writing down the numerical words (or sequences of numbers) of the underlying denotation system in the prescribed order (cf. LOCKE*'s 'Essay':* II, 16, § 7).

In order to prove that human knowledge is not based on innate ideas, but exclusively on sense-experience and reflection, LOCKE also discusses an example from arithmetic. He says (Book I, Ch. 2, § 16) that a child recognizes that $4 + 3 = 7$, not because it is an innate idea (as DESCARTES claimed), or because it remembers prenatal knowledge à la PLATO, or because a god has revealed it to it (as AUGUSTINUS taught it), but rather "*because it has learned to count up to three, up to four and up to seven*."

That sounds amazingly simple, but is it really as simple as LOCKE puts it?

Each child can, if it is to calculate the sum $n+m$ of two numbers n and m, place a series of m dashes after a series of n dashes. To do this, it must first be able to count up to n, and then also up to m. Once it has learned the algorithm for forming the names of all numbers, it can count the series of the dashes altogether and pronounce the result.

We gather from this discussion that LOCKE is quite right. But the procedure of addition is not that simple, because one can only perform addition if one has learned to apply the usual algorithm of addition in the decimal system. If one wants to solve problems concerning addition, one must be able to follow rules. The existence of an equation, e.g., $4 + 3 = 7$, indicates the result of applying the rules that are relevant here. Hence, the validity

of the judgement *4 + 3 = 7* can only be checked by means of the intellect, i.e., by checking whether the rules have been applied correctly. Such a judgment cannot be verified with the senses alone! The elementary arithmetical judgments are therefore not judgments based only on sense-perception (empirically). They are also based on intellectual reflection. - However, as an objection to DESCARTES theory of innate ideas, LOCKE's argument is certainly convincing.

LEIBNIZ complained, in his *'Nouveaux Essais sur l'Entendement Humain'* (written in the years 1703-1705, published posthumously in 1765), that LOCKE did not say anything about the introduction of negative numbers, real numbers and complex numbers (op. cit., book II, Ch. 16). There is much more to lament, for example, that LOCKE says nothing about the introduction of addition, subtraction and multiplication in the realm of whole numbers. The meanings of these operations should also be explained, in particular, the meaning of $(-1) \cdot (-1)$, if it has an extra-mathematical meaning at all.

9.5 LOCKE's comments on some geometrical theorems

Unfortunately, LOCKE did not say anything about the foundations of geometry. But he did occasionally discuss some propositions from EUCLID's *'Elements'*, for example, theorem I.16. We will report on this below.

LOCKE makes a general distinction between propositions that expand our knowledge (i.e., in which more is enunciated than is contained in the definitions of the concepts that appear in the formulation of the proposition) and those that do not. Examples of statements that do not extend our knowledge can easily be given, e.g., statements of the form *'A = A'* (i.e., *A* is *A*), and the like. It is much more difficult to give statements that expand our knowledge.

As an example, LOCKE (IV, 8, § 8) mentions a simple geometric theorem, namely, the so-called *'exterior angle theorem'*, which states that, *in any triangle, if one of the sides is produced, the exterior angle is greater than either of the interior and opposite angles.* EUCLID proves this theorem in the first book of his *'Elements'* (in §16). LOCKE claims that the relationship between the angles expressed in that theorem is not contained in the concept of a triangle and that, therefore, this theorem genuinely expands knowledge, i.e., says more than is contained in the definition of the concept of a triangle.

The relationship between the sizes of the angles in a triangle, which is mentioned in the *'exterior angle theorem'*, is, of course, not explicitly found in the usual definition of a triangle, but it could be that this relationship can nevertheless be logically deduced from the definition. But LOCKE claims that the theorem mentioned (EUCLID, I.16) cannot be deduced from the mere definition (the *'definitio essentialis'*) of the concept of a triangle in a purely logical way. This assertion is somewhat bold, since he could not prove it.

However, we should note that it was able to be proven in the second half of the 19th century. Indeed, in his habilitation lecture in Göttingen on June 10, 1857, BERNHARD RIEMANN presented a model of elliptic, non-Euclidean geometry in which the aforementioned 'exterior angle theorem' (EUCLID, I.16) is not universally true. In this lecture, entitled *'Über die Hypothesen, welche der Geometrie zugrunde liegen'*, RIEMANN designed a geometry of space in which space is immeasurably large (in the sense of the

underlying measure), but not infinite (in terms of expansion), op. cit., III, § 2. In this geometry, 'straight lines' are certain lines that return into themselves. Every two 'straight lines' that intersect at one point also intersect at a second point. The ninth Euclidean 'common notion' (see Chapter 4, Sections 4.1 & 4.4) is therefore not valid in this 'elliptical geometry'. It can be shown that even the 'exterior angle theorem' is not universally valid there (cf. also HEATH, op. cit., Vol. I, p. 280). From this follows the validity of LOCKE*'s* remark. (The 'exterior angle theorem' cannot be proved in 'absolute geometry', and therefore also not from the mere definition of a triangle.)

LOCKE*'s* remark says (in Aristotelian terminology) that the proof of the 'exterior angle theorem' as given in the Euclidean *'Elements'* is not a *demonstratio propter quid* (i.e., does not establish *'why'* (διότι, cur, quare) the alleged fact is true, but only that the alleged fact is true under the assumption of all Euclidean postulates. EUCLID's proof is not based on definitions of essence alone, hence, it is not based on 'reasons' ('causes', in the sense of ARISTOTLE), and therefore is not a 'scientific proof' (again: in the sense of ARISTOTLE) - cf. Chapters 3 & 7.

LOCKE*'s* remark also says (in the terminology of KANT, cf. Chapter 12) that the 'exterior-angle theorem' is *not an analytical judgment.* Neither is the 'triangle sum theorem' (EUCLID, **I**.32) an *analytical judgment*, as we have already seen in Chapter 7, since, here, too, the proof of the theorem must use the axiomatically required properties of the system of all points, all lines and all circles. We will come back to the problems that have become apparent here in the discussion of KANT's conception of mathematics in Chapter 12.

9.6 Psychologism in LOCKE*'s* work

LOCKE made brief comments on a few mathematical theorems, but, unfortunately, did not comment on the problems of the foundation of mathematics. However, we can take some hints from his general program. It can be seen here that - contrary to traditional views - he was of the opinion that the human soul (the psyche, ψυχή) was the creator of mathematical objects. LOCKE was one of the first to interpret the creation of mathematical objects as a psychic event. His only forerunners are PROKLUS, NIKOLAUS CUSANUS, ALEXANDER PICCOLOMINI and probably a very few others (cf. Chapters 7 and 13).

We recall that PLATO believed that the objects of mathematics did not belong to the sense-perceptible world, but to a world of their own, an extramundane realm of objects, which was only mentally perceptible (cf. Chapt. 2). This world of ideas is presented to us, and not our creation. For the recognition of the facts that are true in this world of ideas, it requires recollection (ἀνάμνησις), thinking (διάνοια) and also mental perception (νόησις).

We also recall that ARISTOTLE believed that geometric objects are the sense-perceivable objects of the real world, which become mathematical objects only via 'abstraction and separation' (ἀφαίρεσις and χωρισμός). In the process of 'abstraction and separation', however, no new object is created, because 'abstraction and separation' only takes place in our thoughts and in our speech.

Finally, we also remember the views of the Stoics and the early Christian philosophers (especially AUGUSTINUS), who considered the objects of mathematics to be thoughts of a deity. In this view, the truth of a mathematical theorem can only be recognized if a higher

Being shows it to us (cf. Chapter 8).

With LOCKE, we find a completely different view, because it is argued here that mathematical objects are 'ideas', which *the human soul constructs* via mental reflection and, in particular, also via *'psycological hypostasis of the general'*. In the following two hundred years, this doctrine of LOCKE*'s* became the generally held view, especially because LEONHARD EULER also adopted and propagated LOCKE*'s* view in his *'Lettres à une Princesse d'Allemagne'* (op. cit.), especially in the 100th letter. However, this psychologistic interpretation of the sense in which mathematical objects exist has its major weaknesses. These weaknesses only became apparent, however, when, towards the end of the 19th century, the foundations of mathematics were reconsidered. The psychologistic interpretation disappeared from the scene during this period and is no longer represented anywhere today.

9.7 Discussion

Although the discussion of mathematical problems in LOCKE*'s* Œuvre is not very prolific, LOCKE*'s* critical view has led to some important innovations.

(a) In Chapter 8, we came to the conclusion that the 'where from' of the known in mathematics from antiquity up to DESCARTES was not regarded as a creative contribution of the human mind. But LOCKE was one of the first to contradict these convictions and to claim and prove that the recognition of a truth like $4 + 3 = 7$ was a creative achievement of the human mind. From today's point of view, this assertion may seem like a harmless trifle, but, in light of the two thousand-year-old tradition from PLATO to AUGUSTINUS up to DESCARTES, it is a significant break with a handed-down belief.

(b) ARISTOTLE had not allowed mathematics the freedom to go beyond what is sense-perceivable in his philosophy. It is therefore remarkable that LOCKE dared to take this step beyond ARISTOTLE. He extended the range of mathematical objects from the ideas of sensation (which refer only to the sense-perceivable objects of the real world) to the range of things that can be constructed through mental activity. He allowed for the idea of infinity, and thus also allowed for an 'Analysis infinitorum', as it was developed during LOCKE*'s* lifetime (LEIBNIZ, NEWTON and others). Where LOCKE sees the boundaries to 'wild fantasy' is not so clear, however.

(c) With the question of how great the scope of our knowledge *a priori* is, LOCKE opened up a new field of philosophical investigation, in which GOTTFRIED WILHELM LEIBNIZ, DAVID HUME, IMMANUEL KANT and others were later to have a particular influence.

(d) After all, it was LOCKE who brought *empiricism* and *psychologism* into the discussion of the foundations of mathematics. As we shall see, this was not a happy act, but it was an act that dominated the discussion throughout the 18th and 19th centuries.

(e) LOCKE'*s* contributions to the philosophy of mathematics were not very well thought out. They were quite superficial, but, nevertheless, had a great impact. In the end, they did not prevail, but they did initiate research, and thus greatly promoted development.

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Chapter 10
Rationalism

The question of whether the senses or the mind lead us to true knowledge is very old; it has been asked again and again throughout history and answered in very different ways. In *empiricism,* the primacy of sensory perception over thinking is claimed, and in *rationalism,* conversely, the primacy of thinking over sensory perception. That which is *true* in arithmetic and geometry can, according to the empiricists, be found through sense-perception or, according to the rationalists, in one's own mind. In this and the following chapter, we will discuss some of the firm beliefs of these two positions.

Here, in Chapter 10, however, we want to deal with only one aspect of rationalist thinking, namely, the thesis that it would be possible to construct arithmetic and geometry *without any preconditions*, i.e., without using principles (axioms or postulates) that do not already belong to pure logic.[1] It is claimed, for example, by GOTTFRIED WILHELM LEIBNIZ (see below), that, for this purpose, it is only necessary to give meaningful definitions of the basic terms. Then, all the true statements in these disciplines could be brought about solely on the basis of definitions created through thinking and digging into one's own consciousness, into one's own mind. According to this *'rationalist position'*, the only source from which we draw when we prove theorems of arithmetic or geometry would be the mind (the *'rational mind'*, the *intellect*).

When constructing a mathematical theory, one usually has to say in the outset which kind of objects one wants to study. The construction of a mathematical theory should therefore at least begin with a set of definitions (the definitions of the various kinds of objects), accompanied by - if necessary - a set of suitable postulates (or axioms).

However, if the point of view of *'rationalism'* is adopted, then a mathematical theory should be constructed in such a way that the only source from which we can draw, when we want to prove propositions or theorems, is our own mind. Accordingly, proofs must be *unconditional* and the theories cannot be based on systems of postulates or axioms. This means that the fundament of a mathematical theory should consist only of definitions of basic concepts, and nothing else. In such a theory, all proven theorems will be *apodictically certain.*

[1] Since the principles of pure logic are valid in all sciences (cf. Chapter 3), they are not counted among the presupposed assumptions of any particular theory.

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The question arises: is it possible to construct arithmetic and geometry from the point of view of rationalism?

This question was intensively discussed in the second half of the 17th century (following the discussion about the status of mathematical theories as sciences in the Aristotelian sense, cf. Chapter 7). The following people became involved in this discussion:

> THOMAS HOBBES (1588-1679), ISAAC BARROW (1630-1677), GOTTFRIED WILHELM LEIBNIZ (1646-1716), EHRENFRIED WALTER VON TSCHIRNHAUS (1651-1708), CHRISTIAN WOLFF (1679-1754) and a few others.

They were all convinced that it is possible to construct classical mathematical theories on the basis of definitions alone, without the support of any postulates, axioms or any other sort of hypothesis. However, the usual *'definitiones essentiales'* (*definitions of essence*), which only attempt to specify all of the essential properties of the things to be defined, are not sufficient, as we have already seen in Chapter 7. But they thought, and also hoped, that it could be possible to be successful using another kind of definition, namely, the so-called *'genetic definitions'.* In a genetic definition of a concept, it is indicated how those things that fall under the concept can be generated (or produced). Genetic definitions are sometimes also called *causal definitions*.

We want to report on this, and, in doing so, we first have to talk about the problem of the definability of the basic concepts of geometry in general.

10.1 The problem of definitions in geometry

In some mathematical theories, it seems to be very difficult to give meaningful definitions of the basic concepts. For example, how can we define the terms *'point', 'line', 'straight line'* in geometry, or the terms *'number', 'imaginary number'*, etc., in arithmetic, or the term *'set'* in set theory? Is it permissible to formulate the sought-after definitions in the familiar colloquial language (with its usual semantics), or is it permissible to use, in the definitions, only terms that belong to the specific language of the theory and that have been clearly defined in advance?

In EUCLID's *'Elements',* for example, it says that *'a line is length without breadth'.* Does this really say what a line is, and do such *'lines'* really exist somewhere? - According to this definition, a line would be the *'reification'* (hypostasis, Verdinglichung) of the concept of length. The question arises as to whether EUCLID really deals with such objects in his *'Elements'.* Could EUCLID have defined the concept of a line differently? How could he have defined the somewhat more difficult concept of the *'straight line'* (cf. Chapter 4) correctly?

HERON (Ἥρων), who probably lived in Alexandria in the first century of our era, gave, in his memoir, which has the simple title *'Definitions'*, the vivid description that a *line* is the path of a point when moving, i.e., *'the flux of a point'*. This 'definition' was also given earlier by ARISTOTLE (*'De anima'* I,4, 409a5). This is a description in colloquial language based on the undefined concept of movement (in geometry), and is therefore problematic.

In his commentary on the first book of EUCLID*'s 'Elements'*, PROCLUS defined a *straight line* as being created by the *'rectified and undeflected flow of a point'* (op. cit., p. 292, 296).

PROCLUS emphasizes that this does not mean physical movement, but only an imagined movement (op. cit., p. 296).

But all of these definitions are problematic, because it is not clear what a *'point'* is, and especially how the process of steady rectified flow is to be understood. It seems that exact and conceptually simple definitions of the basic geometric concepts are not available.

10.2 On refraining from defining the basic concepts (DESCARTES, PASCAL, ARNAULD)

RENÉ DESCARTES (1596-1650) and BLAISE PASCAL (1623-1662) thought that it was not necessary to look for exact and meaningful definitions of the basic concepts of geometry, since these concepts were well known and familiar to all people - at least, from daily life.

DESCARTES was convinced that the basic concepts of arithmetic and geometry had not come to us through the senses, but were present in our souls from birth on as ideas, and were therefore known to us. They are *'innate ideas'* (cf. Chapter 8) and, hence, need not be defined.

PASCAL was also convinced of the indefinability of fundamental mathematical concepts. In an essay written around 1655-1658 that remained unfinished (it was published posthumously), conventionally entitled *'De L'Esprit Géométrique'*, he outlined his proposals for a consistent development of arithmetic and geometry.

PASCAL thought that, in a mathematical discipline, one should first of all seek those concepts that are immediately understandable to everyone, even without any attempts to precisely define them. He called these terms *'mots primitifs'.* Examples of such 'primitive terms' are, in his opinion: *Space, time, movement, number, equality*, etc. He considered them to be trans-subjective, self-understood terms that are immediately comprehensible by everyone capable of speaking a language (cf. J.-P. SCHOBINGER 1974, op. cit.). This kind of intuitive awareness lies beyond the rational mind, and is, in PASCAL's words, a *"sentiment du coeur",* i.e., a kind of natural knowledge of the principles of the heart. It seems that, here, the 'heart' plays the role of the Platonic 'nous' (νοῦς).

On the basis of the *'mots primitifs'*, all other terms of mathematics can be introduced by purely nominal definitions. The concept of the number is regarded as a *'mot primitif',* and, hence, remains undefined. The derived concepts such as 'even number', 'prime number', etc., are defined, and correctly termed *'nominal definitions'.*

This shows that the difficult problem of introducing the basic concepts of geometry and arithmetic by suitable definitions was circumvented by PASCAL with the somewhat audacious remark that these basic concepts cannot be defined and are familiar to all people anyway.

In his time, PASCAL correctly estimated the problems that arise with the introduction of the primitive basic concepts. However, he erred in thinking that the whole content of primitive concepts (the *'mots primitifs'*) is given to us by intuition and would be revealed to us by a 'natural insight' (*la lumière naturelle*). What is the 'natural insight' of which PASCAL speaks, and how can it be controlled? PASCAL gives no answer to this question.

It is evident that PASCAL*'s* ideas are committed to the Patristic (especially AUGUSTINUS) and the Platonic-New-Platonic Renaissance movements, which we discussed in Chapter 8.

Neither Neoplatonism nor the Patristic movement aimed at a thorough analysis of the foundations of mathematics. For his part, PASCAL neither provided such an analysis, nor did he try to.

ANTOINE ARNAULD [2] based his books *'La Logique, ou L'Art de Penser'* (Paris 1662) and *'Nouveaux Élémens de Géometrie'* (Paris 1667, Den Haag 1690) on PASCAL's views, and thus contributed to their dissemination.

Entirely in the spirit of ARISTOTLE, ARNAULD assumes that the objects of arithmetic and geometry are the things of the natural real world, which are only considered in terms of size, number, extent and form. In this respect, every human being has knowledge of distance, length, number, etc., from their everyday dealings with the things of the world. He therefore expressly refrains from defining the terms *'straight lines', 'area'*, etc., and thinks, for example, that it is sufficient to say that the ends of lines are called *'points'*. He writes, in his *'Nouveaux Elemens de Geometrie',* Livre V, (op. cit. p. 146):

> *"Les idées d'une surface plate et d'une ligne droite sont si simples, qu'on ne feroit qu'embrouiller ces termes en les voulant definir. On peut seulement en donner des exemples pour en fixer l'idée aux termes de chaque langue."*
>
> [The concepts of a flat surface and a straight line are so simple, that one would only confuse these terms by wanting to define them. One can only give examples to indicate the (respective) concept with colloquial words.]

ARNAULD seems to agree with PASCAL that the basic concepts of a mathematical theory need not be explicitly defined, and that it is sufficient to indicate what is meant by individual examples.

PASCAL*'s* essay *'De L'Esprit Géométrique'* had an enormous impact on many mathematicians. Even in the 20th century, one can still find followers of his method. For example, EMIL BOREL (1871-1956) writes, in his *'Leçons sur les fonctions de Variables réelles'* (Paris, 1905, p. 1), about the fundamental concept of sets:

> *"L'idée d'ensemble est une notion primitive dont nous ne donnerons pas de définition. Citons seulement quelques exemples d'ensembles: l'ensemble des points d'une droite,..*(etc.)"
>
> [The idea of *'set'* is a primitive notion for which we will not give a definition. Let's just give a few examples of sets: the set of points on a straight line (,... etc.).]

FELIX HAUSDORFF (1868-1942) also adopted PASCAL's and BOREL's point of view in his works on set theory. For example, he writes, in his book on *'Mengenlehre'* (de Gruyter-Verlag, Berlin-Leipzig, 1927), right at the beginning:

> *"Eine Menge entsteht durch Zusammenfassung von Einzeldingen zu einem Ganzen. Eine Menge ist eine Vielheit, als Einheit gedacht. Wenn diese oder ähnliche Sätze Definitionen*

[2] ANTOINE ARNAULD was born on February 16, 1612, in Paris. He received his doctorate in the field of theology in 1641. From 1643 to 1656, he taught at his hometown's famous Sorbonne. Because of his religious convictions, he had to leave the University of Paris. On June 17, 1679, he fled from France to the 'Fürstbistum' of Lüttich (in the 'Holy Roman Empire of German Nations'), and remained there until his death, on August 8, 1694.

> *sein wollten, so würde man mit Recht einwenden, daß sie* idem per idem *oder gar* obscurum per obscurius *definieren. Wir können sie aber als Demonstrationen gelten lassen, als Verweisungen auf einen primitiven, allen Menschen vertrauten Denkakt, der einer Auflösung in noch ursprünglichere Akte vielleicht weder fähig noch bedürftig ist."*
>
> [A *'set'* is created by combining individual things into a whole. A set is a multitude, conceived as a unit. If these or similar assertions wanted to be definitions, one would rightly object that they define *idem per idem* or even *obscurum per obscurius*. We can, however, read them as demonstrations, as references to a primitive act of thought familiar to all people, which is perhaps neither capable of nor in need of dissolution into even more original acts.]

It is unsatisfactory, however, to pretend, where definitions need to be given, that they are not necessary. The problem of how to control this *'primitive act of thinking familiar to all people'* remains unsolved. But such a control is necessary if one wants to avoid the well-known antinomies of set theory.

10.3 The attempt to define the basic concepts with *'genetic definitions'*

From which sources do we draw when we prove mathematical theorems? This is the question we asked ourselves at the beginning of this book, but we have still not found a satisfactory answer. Even the answers given by DESCARTES, PASCAL and ARNAULD in the case of the definitions of the fundamental concepts of geometry cannot satisfy us, as we have seen. They argued that it is not always necessary to give precise definitions for the basic concepts of a mathematical theory.

If one really wants to announce the sources, one must begin with clear and meaningful definitions of the basic concepts (if that is possible) and then name all of the axioms or postulates upon which one wants to base proofs.

The intense discussions that took place in the sixteenth and early seventeenth centuries on the status of geometry gradually led to considerations as to whether it might be advisable to replace the well-known, but unfortunate, Euclidean definitions with definitions that describe the bringing into existence, i.e., the *'generation',* of the objects in question. Such *'genetic'* definitions were found in the afore-mentioned memoire by HERON, *'Definitions'* (op. cit.). We have already mentioned how HERON defines the concept of the line.

For the concept of the *circle,* for example, HERON first gave the Euclidean definition that a circle is the geometric locus of all points at a same distance from a fixed point, and then the following *causal* (or *genetic*) *definition* that

> *a circle* is created *when a line, by remaining in the same plane while one end point is fixed, is moved around with the other until it is brought back to the same position from where it started to move.*

The definition given by EUCLID says nothing as to the existence or non-existence of the thing defined. The definition given by HERON, however, has a genetic character, since it contains a description of a method for constructing it.

CLAVIUS, in his annotated edition of EUCLID*'s 'Elements'* (Rome 1574; Cologne 1591), referred to HERON*'s* definitions, which thus had the effect that, in the following decades, several authors tried to replace the classical definitions of EUCLID with the (perhaps?) more

meaningful genetic definitions. This gradually led to a new understanding of the axiomatic method.

10.4 The contributions of HOBBES (1655) and BARROW (1664)

Such a new conception of the axiomatic method can be found, by way of suggestion, in the first section, entitled *'De Corpore',* of the *'Elements of Philosophy',* written by THOMAS HOBBES (1588-1679) and published 1655 in London. In this work, HOBBES wanted to show that a certain form of EUCLID*'s* axiomatic method is also applicable in philosophy. He called his work *'Elementa Philosophiae'* ('Elements of Philosophy'), in deliberate allusion to EUCLID*'s 'Elements' (of geometry and arithmetic).* For this purpose, he first dealt with Euclidean geometry.

HOBBES suggested that, in geometry, those objects *'which have a cause and a mode of generation'* should be defined by these causes or modes of generation (HOBBES, op. cit., p. 90). For the 'straight lines' and the 'circles', he quoted HERON's definitions (see above): these objects are created by the flow of points. HOBBES also mentioned that, in such an accomplishment of geometry, the *'starting points of proofs are the definitions'* (op. cit., p. 89). There is no need to make postulates.

In two further extensive papers entitled *'Examinatio et Emendatio Mathematicae Hodiernae*' (cf. HOBBES*: 'Opera Philosophica quae latine scripsit',* vol. 4, London 1845) and *'Six Lessons to the Professors of Mathematics'* (HOBBES*: 'The English Works*', vol. 7, London 1845), HOBBES made some additions. But even from them, it is not clear whether the program that he had in mind for geometry is feasible.

However, the program that he had in mind for philosophy became clearer. According to his view, philosophy deals with *'the order, causes and effects of things in the world'*, as he wrote in the *'Introduction: To the reader'* of his work *'Elements of Philosophy'.* For this purpose, the things to be thought about must be introduced with genetic definitions in order to be able to progress (in a rational way) from the recognized causes to the effects. It would not be necessary to start from postulates. - The program is ambitious, but remained fragmentary, and was never carried out.

ISAAC BARROW, NEWTON*'s* teacher at Cambridge, also addressed the problem of definitions in his *'Lectiones Mathematicae'.* These lectures in mathematics had been presented by BARROW in Cambridge in the years 1664-1666, but they were not published until 1683 (in London), and thus only become effective from then on.

In his *'Lectiones Geometricae'* (London 1670), he advocated the Aristotelian view that geometry was about the sense-perceptible sizes of objects in the real world and the sense-perceptible flow of time. Accordingly, he moved the subject area of geometry into the area of sense-perceivable things. Curves are given by the flow of points. With BARROW, the composition of algebraic curves, which DESCARTES treated in his *'Géométrie'* (1637), became compositions of movements (op. cit., Lectio III, p. 185 ff.). BARROW, like HOBBES, was of the opinion that the propositions of geometry should be deduced from definitions alone (op. cit., *Lectio Math.* VII, pp 106-107), but he did not want to go into detail on new foundations of geometry in which only (causal) definitions and no postulates appear. - It was LEIBNIZ who began to work on such a new kind of founding geometry.

10.5 The contribution of LEIBNIZ (ca. 1676)

GOTTFRIED WILHELM LEIBNIZ (1646-1716) was - like DESCARTES (cf. Chapter 8) - convinced

> *"... que toute l'Arithmétique et toute la Geometrie sont innées et sont en nous d'une maniere virtuelle, en sorte qu'on les y peut trouver en considerant attentivement et rangeant ce qu'on a déja dans l'esprit, sans se servir d'aucune verité apprise par l'experience ou par la tradition d'autruy, comme Platon l'a monstré ....*
>
> [(...) that all arithmetic and all geometry are innate and potentially within us in such a way that, if we consider and order what is already present in the mind, we can find it in it without resorting to any truth that we have learned through experience or tradition (...)]
>
> LEIBNIZ, *'Nouveaux Essais',* book I, Chapter 1, §5ff, op. cit., p. 22.

In this respect, LEIBNIZ, like DESCARTES, took the position of *rationalism* that arithmetic and geometric truths could be drawn entirely from our *ratio* (our spirit, soul, or rational mind) without having to rely on sensory impressions. But they took this position in very different ways.

We have already pointed out in Chapter 8 that DESCARTES believed that, by looking directly at the ideas that are innate to us, we could also recognize the *mathematical truths.* He called this kind of mental perception *'intuition'.*

LEIBNIZ, on the other hand, did not want to rely on intuition to discover mathematical truths. He was convinced that, in order to practise a mathematical discipline, it would only be necessary to know the definitions of the respective basic concepts.[3] In his *'Nouveaux Essais'* (op. cit., p. 544, line 9), he wrote that neither in arithmetic nor in geometry was it necessary to use postulates (or axioms) in the proofs. It would be possible in these disciplines to derive everything from definitions and *'identical propositions'.* The propositions that LEIBNIZ called 'identical propositions' (but that were not precisely defined) were later called *'universally valid'*[4] (*'logisch allgemeingültig'* by PAUL BERNAYS and MOSES SCHÖNFINKEL, Math. Annalen 99 (1928), pp. 342-372). These are statements

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[3] *"Nonne definitio est principium demonstrationis?"* [Isn't definition the principle of every demonstration?] - says LEIBNIZ*'s 'Dialogus de connexione inter res et verba*' - see LEIBNIZ*'s 'Philos. Schriften'*, Volume IV, Edition H. HERRING, op. cit., p. 28/29.

[4] LUDWIG WITTGENSTEIN called them *tautologies* in his *'Tractatus logico philosophicus'* (1918, # 4.26). The word comes from the Greek ταὐτὸ λέγειν (= saying the same thing twice). If one writes a statement Φ of propositional logic in its conjunctive normal form $\Psi_1 \wedge \Psi_2 \wedge \ldots \wedge \Psi_n$, then Φ is, according to a theorem of PAUL BERNAYS (1918, published in Math. Zeitschr. 25 (1925), pp. 305-320), a tautology exactly when each conjunctive member Ψ_i has the form $(A_{k_i} \vee \neg A_{k_i}) \vee \Gamma_i$, that is, $(A_{k_i} \rightarrow A_{k_i}) \vee \Gamma_i$. One can see here very clearly that, in tautologies, the *same thing* is said *twice*, namely, A_{k_i}.

For an 'identical theorem', LEIBNIZ gave the example: *An equilateral triangle is a triangle.* He noted identical propositions in the form *A = A*. He thought that a statement is 'identical' if the predicate is contained in the subject (cf. also HOBBES, *'De Corpore'* I,4, op. cit., p. 48, and LEIBNIZ*' 'Discours de Métaphysique'*, op. cit.). In his *'Nouveaux Essais',* p. 420, LEIBNIZ writes: *'The original truths of reason are those which I call ... identical, because they only seem to repeat the same thing, without lecturing us.'* One may therefore identify them with the tautologies.

that, according to LEIBNIZ, are all innate to us.

10.6 LEIBNIZ's *'Dialogue for an Introduction to Arithmetic and Algebra'* (ca. 1676)

LEIBNIZ tried to show, in his *'Dialogue for the Introduction to Arithmetic and Algebra'*, that, in arithmetic and algebra, it is possible to prove all of the theorems on the basis of some definitions and some 'identical statements' (tautologies) alone. Special postulates or axioms are - in LEIBNIZ's opinion - not necessary here.[5] This *'Dialogue'*, which he wrote ca. 1676, but which remained unpublished and only appeared posthumously three hundred years later in 1976, was translated and edited by EBERHARD KNOBLOCH (op. cit.). He writes, in his epilogue (p. 185), that LEIBNIZ wanted to use this piece to demonstrate the fundamental correctness of the Platonic epistemology and theory of recollection, at least, for arithmetic. LEIBNIZ himself expressed it, in his *'Discours de Métaphysique'* (Metaphysical treatise, § 26 (1686), op. cit.), quite similarly:

> *"Et rien ne nous sçauroit estre appris, dont nous n'ayons déja dans l'esprit l'idée qui est comme la matiere dont cette pensée se forme. C'est ce que Platon a excellement bien consideré, quand il a mis en avant sa reminiscence qui a beaucoup de solidité, pourveu qu'on la prenne bien qu'on la purge de l'erreur de la preexistence et qu'on ne s'imagine point que l'ame doit déja avoir sçeu pensé distinctement autres fois ce qu'elle apprend et pense maintenant. Aussi at-il confirmé son sentiment par une belle experience, introduisant un petit garçon qu'il mene insensiblement à des verités tres difficiles de la Geometrie touchant les incommensurables, sans luy rien apprendre, en faisant seulement des demandes par ordre et à propos. Ce qui fait voir que nostre ame sçait tout cela virtuellement, et n'a besoin que animadversion pour connoistre les verités et par consequent qu'elle a au moins les idées dont ces verités dependent."*
>
> [And nothing can be taught to us whose idea we do not already have in our minds, which is, so to speak, the matter from which this thought is formed. PLATO considered this very well when he presented his doctrine of reminiscence (anamnesis: *'Phaidon'* 72e-75f), which is well founded, provided that one understands it correctly, that one purges it of the error of pre-existence and does not imagine that the soul must have known and thought *clearly and distinctly* earlier what it is currently learning and thinking. He has also confirmed his view by a beautiful experience, introducing a young boy whom he leads imperceptibly to the very difficult *truths* of geometry concerning incommensurable quantities, without teaching him anything, only by asking him questions in an orderly and appropriate manner (*'Menon'*, 82a-85b). This shows that our soul knows all this virtually, and that it only needs to be attentive to know the *truths,* and that it therefore has in any case those ideas on which these truths depend].

LEIBNIZ thus thought that our knowledge of the properties of mathematical objects is innate, and that we could bring this knowledge into consciousness *'by digging'* (*'Nouveaux Essais',* op. cit., p. 9) as we grew up.

The *'Dialogue for Introduction to Arithmetic and Algebra'* has a similar structure to the Platonic *'Menon'*. In the LEIBNIZ dialogue, the character of Charinus, a stand-in for Leibniz

[5] See also the LEIBNIZ essay *'Prima Calculi Magnitudinum Elementa demonstrata in Additione et Subtractione, usque pro ipsis Signorum + et -'*, In *'*LEIBNIZ *math. Schriften'* (edited by GERHARDT), volume 7 (Halle, 1863), pp. 77-82.

himself, conducts a conversation with the 'little son of Aretaeus', who *'has not yet learned the art of arithmetic'* (op. cit., p. 12/13). Charinus, the leader of the conversation, wants to show that the 'little son' need only get to know the wording of some definitions in order to be able to bring the whole art of arithmetic (so to speak, *'by digging'*) from within himself into his consciousness. Charinus says that he

> *would not put forward to the boy certain new truths or doctrines, but would only give him new names and signs that are suitable for expressing thoughts.* (*'Dialogue'*, op. cit., p. 26/27.)

Over the course of the conversation, the subject covers calculation with letters (à la VIÈTE), addition, multiplication, exponentiation, subtraction and calculating with imaginary and complex numbers.

In the case of *addition*, only the colloquial meaning of the word *'collecting'* is recalled *("Ut enim duas res in unum locum conjicere est addere,..."*, loc. cit., p. 38/39). The boy can immediately see from this definition of 'addition' (allegedly without further instruction) that addition is commutative: $a + b = b + a$. Charinus states that this law was in the boy's mind, but still disordered, and only had to be elicited *'from the treasure of his soul' ("ex tuae ipsius animae thesauro"*) (loc. cit., p. 36/37 and 48/49).

We have to object to this somewhat rash argument that the addition of *cardinal numbers* a and b is defined solely by *'collecting'*. Such an addition of cardinal numbers is well defined and commutative. This can easily be proved with some set-theoretical concepts (considering the union of two disjoint finite sets).

However, the *numbers* were introduced on pages 16-27 (loc. cit.) as *ordinal numbers*, since they were identified with the *recursively* defined numerals (or representations of numbers in the decimal system). The addition (and also the multiplication) of ordinal numbers would thus also have to be introduced through recursion, and the proof of the commutativity of the addition (and also of the multiplication) would have to be proved through *'complete induction'* (as RICHARD DEDEKIND put it in his famous 1888 booklet *'Was sind und was sollen die Zahlen?'*) – So, LEIBNIZ's argument is not conclusive here.

For LEIBNIZ, arithmetic is constructed free of any empiricism. Empiricism has indeed been eliminated, but it has been shown that the laws of arithmetic cannot be 'dug out' from the rational mind (the *ratio*) alone by using only definitions. The presupposition of certain postulates remains indispensable.

The same can be said about the structure of geometry based solely on genetic definitions, which LEIBNIZ worked out in his treatise *'In Euclidis Πρωτα'* (posthumously published in: *'Math. Schriften'*, edited by C.I. GERHARDT, Volume 5, op. cit., pp. 183-211). We do not want to go into this, and so turn instead to LEIBNIZ*'s* attempt to prove the axioms of equality from a sufficiently strong definition of the concept of equality.

10.7 Proof of the axioms of equality

In a short essay, LEIBNIZ tried to show that the axioms of equality, which are part of logic, can be proved on the basis of suitable definitions alone. His results are contained in the following short paper:

G.W. LEIBNIZ: *'Prima calculi magnitudinum elementa demonstrata in additione et*

subtractione, usuque pro ipsis signorum + et -' (posthumously published in: *'Math. Schriften'*, Volume 7, pp. 77-82).

LEIBNIZ only considers the *equality* of objects with respect to magnitude (i.e., size). For EUCLID, e.g., two lines are equal if they are congruent, and DAVID HUME defines two finite collections of objects as equal with respect to their size, *'when their elements can be so combined that each element of the first collection corresponds to exactly one element in the other collection'*. LEIBNIZ, however, defines equality with respect to size in a uniform way for all kinds of objects as follows:

> Definition. *Aequalia sunt quorum unum alteri substitui potest salva magnitudine. Et ita designatur a=b.*
>
> [Two things *a* and *b* are *equal*, provided *a* can everywhere be replaced by *b* without changing the size. And this will be denoted *a=b.*]

LEIBNIZ shows that reflexivity, symmetry and transitivity of equality (with respect to size) can be drawn from this definition.

Reflexivity: Apparently, every term t is equal (in size) to itself: $t = t$.

Symmetry: We assume $a = b$. Because reflexivity is already clear, we also have $b = b$. Since a equals b, we may substitute a for the second b in the assertion $b = b$ and obtain $b = a$, Q.E.D.

Transitivity: Assume $a = c$ and $b = c$. Since b and c are equal, we can replace c with b in the assertion $a = c$ and obtain the assertion $a = b$, as claimed, Q.E.D.

In a similar way, LEIBNIZ also proves some other axioms of equality, for example, the following sentence, which is already found in EUCLID*'s* first book of *'Elements'* as the second 'common notion':

Theorem: *If equals be added to equals, the wholes are equal* ("*Si aequalibus addas aequalia, fiunt aequalia."*):

$$a = b \,\&\, x = y \Rightarrow a + x = b + y.$$

However, LEIBNIZ*'s* definition of equality is not a definition that can be written down in the language $\mathcal{L}$ of the mathematical theory under consideration. It belongs to higher order logic, since it uses a quantification over all terms of $\mathcal{L}$. It is, hence, not a true definition, but a principle that belongs to the Meta-Theory.

For a detailed discussion of LEIBNIZ*'s* definitions of equality and similar definitions of the *identity of terms* (*"Eadem sunt, quorum unum potest substituted alteri salva veritate"*), we refer to the essay by RAILI KAUPPI (1966), op. cit. and our own essay: U. FELGNER: '*Die Begriffe der Äquivalenz, der Gleichheit und der Identität* (2020)', op. cit.

10.8 The concept of axiomatics in TSCHIRNHAUS (1687)

The new concept of axiomatics, which HOBBES only hinted at with a few strokes of the pen, was systematically further developed by EHRENFRIED WALTER VON TSCHIRNHAUS (1651-1708), in his treatise *'Medicina mentis, sive artis inveniendi praecepta generalia'* (Amsterdam 1687, Leipzig 1695). He called the causal definitions, which indicate how the

things that fall under certain concepts can be produced, *'genetic definitions'*. TSCHIRNHAUS developed a philosophy that he called *'Medicina Mentis'* (i.e., medicine for the mind). In his opinion, to truly understand a subject, one must be able to reproduce it in the mind. The definitions of the basic concepts must therefore also include the methods for the construction or mental reconstruction of objects. In his book *'Medicina Mentis'*, he wrote

> *"Circà definitiones notanda in principio; Definitionem, juxta dicta, esse primum alicujus rei conceptum, seu primum quod de re concipitur. Hinc clarum*
> *Primò, nostri planè esse arbitrii tales à nobis formari conceptûs. ...*
> *Secundò patet, omnem rei singularis definitionem semper ejusdem rei primum formationis modum debere includere, quem alicujus rei Generationem nuncupabo. Rem enim quandam verè concipere nihil aliud est, quam actio seu formatio mentalis alicujus rei; ... omnis legitima seu bona definitio, includet generationem."*
>
> TSCHIRNHAUS: *'Medicina Mentis'*, Amsterdam 1687, p. 49-50.

> [As far as definitions are concerned, it should be noted at the outset that a definition is the basic concept of a thing or the first thing that is understood by it. From this it is clear:
> *firstly,* that such concepts can be formed by us or that it is entirely in our will to establish definitions. …
> *Secondly,* it is obvious that any definition of a single thing must always include the first type of construction of that thing, a type of construction which I will call the *generation* (production) of a thing. For to truly comprehend a thing is nothing else than the activity or the mental process of its formation; ... so, indeed, every proper or good definition will imply a production.]

TSCHIRNHAUS outlines his scientific theory in a few words as follows:

> *"... primò omnes possibiles primos conceptûs, ex quibus formantur reliqui, redigam in ordinem, atque imposterum Definitiones nominabo: secundò has ipsas definitiones in se considerabo, & hinc deductas proprietates appellabo Axiomata: tertiò definitiones inter se jungam omnibus modis, quibus id fieri potest, ac veritates inde derivatas Theoremata dicam."* TSCHIRNHAUS: *'Medicina Mentis'*, Amsterdam 1687, p. 49.

> [... firstly, I will put into order all possible basic concepts (primi conceptus) from which the others are formed, and will later call them *definitions*; secondly, I will consider these definitions in themselves and call the peculiarities derived from them *axioms*; thirdly, I will connect the definitions in every possible way and call the truths derived from them theorems.]

Here, the axiomatic theory consists merely of a list of definitions, namely, the genetic definitions of the basic concepts. From these definitions, one 'distils' (or 'extracts') the particular *'peculiarities'* of the defined things and formulates them as axioms of theory. From these axioms, the theorems of the theory can then (as usual) be obtained through purely logical conclusions.

This is a new conception of axiomatics. In contrast to HOBBES, BARROW, LEIBNIZ and others, theorems are not deduced in a purely logical manner directly from the definitions, but an intermediate step is taken in the following way. Interpretation (and exegesis) of the definitions leads to the formulation of axioms, and formal logical deduction leads from axioms to theorems.

For many - but not all - geometric concepts, genetic definitions have been known since antiquity. However, TSCHIRNHAUS is also unable to say how the concept of a 'point' or the concept of a 'straight line' can be introduced genetically. This shows that, even with the introduction of genetic definitions, not all difficulties that arise in the foundation of geometry can be solved.

10.9 *'The method of teaching Mathematics'* according to CHRISTIAN WOLFF

TSCHIRNHAUS completed the work *'Medicina Mentis'* as early as 1682. The first edition was published in Amsterdam in 1687, and the second edition in Leipzig in 1695. It had a significant influence during those years. CHRISTIAN WOLFF [6] (1679-1754) studied the work carefully and, as we know from his autobiography (1643), had many details explained to him by TSCHIRNHAUS during a meeting in Leipzig in 1705.

WOLFF was strongly influenced by the axiomatic method as presented by TSCHIRNHAUS. He based all of his mathematical textbooks on it. This led to the wide dissemination of this form of axiomatics. In the first volume of his famous *'Anfangsgründe aller Mathematischen Wissenschaften'* (Leipzig, 1710), WOLFF added a *'Kurtzen Unterricht von der Mathematischen Methode oder Lehrart'*. There, he writes that the principles of a mathematical theory are the well-defined basic concepts, and that, from the contents of these basic concepts, the axioms of the theory results (through interpretation) and that, from the axioms, the theorems can be obtained through formal-logical reasoning. This is *'the order which the mathematicians use in their lectures'* (WOLFF, op. cit., § 1).

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[6] CHRISTIAN WOLFF was born in Breslau in 1679. He studied mathematics, physics and theology in Jena and habilitated, in 1703, with a treatise on differential calculus. In 1707, he became professor of mathematics and natural science in Halle/S. In 1710, he began to publish his multi-volume work '*Anfangsgründe aller Mathematischen Wissenschaften'* and, in 1716, he published his famous *'Mathematisches Lexicon'*. In 1720, his treatise *'Vernünftige Gedancken von Gott, der Welt und der Seele des Menschen, auch allen Dingen überhaupt'* was published. At the instigation of Pietist theologians, WOLFF lost his professorship in Halle in 1723, through a cabinet order of the 'Soldier King' FRIEDRICH WILHELM I. He was ordered to leave Halle within 24 hours, and the Prussian states within two days, under threat of strangulation. WOLFF was subsequently accepted at the University of Marburg/Lahn.

However, Crown Prince FREDERICK (that is, FREDERICK II, also known as 'FREDERICK THE GREAT') so appreciated WOLFF's treatise that he had it translated into French and sent to VOLTAIRE. In September 1736, VOLTAIRE thanked him for the book, writing:

> *"Je regarde ses idées métaphysiques comme des choses qui font honneur à l'esprit humain. Ce sont des éclairs au milieu d'une nuit profonde; c'est tout ce qu'on peut espérer, je crois, de la métaphysique."* (Œuvres Complètes de VOLTAIRE, vol. 64, Kehl 1785, p. 16.)
>
> [I consider his metaphysical thoughts to be something to honor the human spirit. They are flashes of lightning in the middle of a deep night; that's all one can hope for, I think, from metaphysics.]

Immediately after FREDERICK'S accession to the throne, WOLFF was rehabilitated in 1740, and called back to Halle. He was glad to return. He died on April 9, 1754.

CHRISTIAN WOLFF had a great influence on the development of philosophy in the 18th century. IMMANUEL KANT called him *"den gewaltigsten Vertreter des Vertrauens in die Macht der Vernunft"* [the most powerful representative of confidence in the power of reason.]

According to the conception of TSCHIRNHAUS and WOLFF, the foundations of a mathematical discipline consist of a set of genetic definitions. The axioms of the discipline are derived from the intuitive contents of these definitions. The axioms are thus *ex terminis* certain, because what is expressed in them must result directly from the contents laid down in the definitions. Notice, that in this way also MORITZ PASCH presented his exposition of Geometry in 1882 - see Chapter 11, section 11.4, pp. 157-158, for some details.

In WOLFF*'s* opinion, it is impossible that two different mathematical theories, which are both based on genetic definitions and nothing else, cannot lead to two contradictory theories. In fact, the reproducibility of the various types of basic objects is possible everywhere in this world, at any time, and always with the same result. The axioms thus result as necessary consequences from the genetic definitions, and are therefore all *apodictically certain*. From the list of axioms, the theorems of the discipline are obtained through logical deduction. *The theorems would then also all be apodictically certain.*

10.10 Discussion

TSCHIRNHAUS' and WOLFF's conception of the axiomatic structure of a mathematical theory remained (at least, in Germany) the predominant view until the end of the 19th century.

IMMANUEL KANT (1724-1804), e.g., incorporated the notion of a *'genetic definition'* in his doctrine of *'pure intuition'* (*"reine Anschauung"*). In fact, as we shall see in Chapter 12, section 12.7, a concept is, according to KANT, *'constructible in pure intuition'* if it has a genetic definition and the construction of the defined object can be carried out in our intuition *'without having borrowed the pattern from any experience'.*

Another prominent example is CANTOR's set theory. CANTOR*'s* famous definition of the *concept of 'set'* was placed in the outset of his great final treatise *'Contributions to the foundations of transfinite set theory'*. The paper is written in German and has the original title: *'Beiträge zur Begründung der transfiniten Mengenlehre'* (1895 & 1897, op. cit.). The definition is as follows:

> *"Unter einer »Menge« verstehen wir jede Zusammenfassung M von bestimmten, wohlunterschiedenen Objekten m unserer Anschauung oder unseres Denkens (welche »Elemente« von M genannt werden) zu einem Ganzen."*
>
> [By a *'set'* we understand each bringing together *M* of determined and well-distinguished objects *m* of our intuition or our thought (which are called the *'elements'* of *M*) into a whole.]

This is obviously an attempt at a genetic definition, because, on the one hand, it is indicated *what sets are* (namely, collections of certain things, where the collection itself is understood as a further object) and, on the other hand, it is indicated *how sets are created*, namely, through an act of aggregation, or, in other words, through an act of bringing together (gathering, 'zusammenfassen').

CANTOR does not start, in his *'Mengenlehre'*, from axioms or postulates. He starts by proving theorems directly after the concept of a *'set'* is thus defined. This is done by exhausting the content of the genetic definition (cf. Chapter 15).

It is well known that CANTOR could not prevent the occurrence of logical contradictions

in his set theory, because the definition of the concept of a *'set'* that he gave is written down in a colloquial language, in which the words do not have a sufficiently precise meaning. In his earlier philosophical papers, he very clearly indicated the sense in which he wanted to use the word *'Zusammenfassung'* in his 'Mengenlehre', and in which cases it should be possible to collect various objects *'into a whole'*. But all of this was not repeated in the formulation of the definition of his concept of a *'set'*. Nowhere in the paper did he explain his important distinction between *'transfinite'* and *'absolute infinite'* multitudes, although, in the title of his paper, he even spoke of a theory of *'transfinite sets'*. This is quite understandable, since the different meanings of the two concepts are explained in terms of theology. Clearly, mathematics should not rely on sources for gaining knowledge that are inaccessible for mathematics.

It thus happened that most of his colleagues understood his definition in the usual colloquial sense and observed (under this reading) that it leads to inconsistencies: namely, the well-known antinomies of set theory, observed by C. BURALI-FORTI (1897), E. ZERMELO (1900/1901), B. RUSSELL (1902/1903) and others. We will come back to this problem in Chapter 15. Here, it will be sufficient to remark that the use of colloquial language in mathematics may lead to severe difficulties. In fact, at the beginning of the 20th century, it led to a severe crisis in the foundations of mathematics. The crisis was surmounted by introducing the so-called *position of formalism.* A levelling-down of the fundaments of mathematics was possible now, and, in particular, an emancipation from the use of colloquial languages by introducing so-called *'formal languages'*. We will come back to this in Chapter 20.

The above discussion shows that the introduction of genetic definitions, together with a new conception of the axiomatic method, was not as successful as it seemed for quite some time. To the old question of how to formulate meaningful definitions in an axiomatic presentation of a mathematical theory, even the new versions of PASCAL, ARNAULD, HOBBES, LEIBNIZ, TSCHIRNHAUS and WOLFF did not provide a satisfactory answer! Thus, *rationalism* was not very successful in solving the fundamental problems of putting mathematics on a safe basis. - In the next Chapter, we will explore whether *empiricism* might be more successful in this respect.

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Chapter 11
Empiricism in Mathematics

Ogni nostra cognizione prencipia da sentimenti.
[Our cognitions all begin with sensations.]
LEONARDO DA VINCI*, Diary 'Trivulzio', 20v.*

Empiricism is the position in philosophy in which *all* knowledge is ultimately derived from sensory experience. Accordingly, all that we humans can know is ultimately based on sensory perception. All questions relating to the origin and justification of our knowledge can ultimately only be decided by referring to our sensory perceptions.

In Greek, ἐμπειρία is experience, the knowledge conveyed by the senses.

Since antiquity, philosophers have always been attracted to empiricism to a greater or lesser extent (ARISTOTLE , LUCRETIUS, PROTAGORAS and others). But, in the 17th, 18th and 19th centuries, especially in England, Ireland and Scotland, a number of philosophers appeared who professed empiricism and who criticized mathematics as it had been practiced since antiquity with unswerving doggedness. These philosophers did not even want to accept large parts of mathematics. In particular, they interfered in the dispute over the new calculations with indivisibilities, infinitesimals and fluxions (BONAVENTURA CAVALIERI, GOTTFRIED WILHELM LEIBNIZ, ISAAC NEWTON and others) and, at the same time, fought against the rationalism that was predominant on the European continent.

Let us see what they had to say, starting with the Irish theologian and philosopher GEORGE BERKELEY.

11.1. BERKELEY*'s* critique

GEORGE BERKELEY (1685-1753), in many of his writings, strongly criticised the ontological views underlying Euclidean geometry and arithmetic on the one hand and LEIBNIZ-NEWTON*ian* infinitesimal calculus on the other. He called for a revision of the foundations of these theories on an empirical basis.

BERKELEY called everything that the mind deals with when thinking - similar to JOHN LOCKE - *'ideas'* (cf. BERKELEY*'s 'Treatise'*, in *'Works',* Vol. 2, § 1, § 4). In BERKELEY*'s* opinion, 'ideas' only come into consciousness through direct *sense-perceptions* and through the mental activities of putting together, dividing and imagining that which is originally sensorially perceived.

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BERKELEY*'s* credo was the dictum of PROTAGORAS: *"esse est percipi"* (as he wrote in 1710, in his *'Treatise',* §§ 2,3,6). In his *'Philosophical Commentaries'*, BERKELEY expressed this conviction with the following words (cf. his *'Works',* Vol. 1, No. 429):

'Existence is either being perceived (percipi) or perceiving (percipere)'.

This is his highest principle, to which all sciences (except theology) must submit.

BERKELEY complained that *'general abstract ideas'* were used in many sciences. In particular, he complained about this in the case of arithmetic. If numbers (as in EUCLID) are to be thought of as sets of units, then the difficulty arises that the words *'unity'* and 'set of units' do not correspond to 'ideas'[1] (in BERKELEY*'s* sense) (cf. BERKELEY*'s Treatise,* § 120). So, if there are no 'units', then there are no abstract numbers either. In arithmetic, BERKELEY believes that there are no objects that can be regarded as numbers. What exist are only signs or figures or words that can be used for counting. He wrote, in his *Treatise,* section 122:

„In Arithmetic therefore, we regard not the things, but the signs, which nevertheless are not regarded for their own sake, but because they direct us, how to act with relation to things, and dispose rightly of them."

For BERKELEY, numbers are not abstract entities, but merely words or figures, such as the sequences of numbers in the Indo-Arabic place-value-system. For him, arithmetic and algebra are mere *'name sciences'* (cf. *'Philosophical Commentaries',* nos. 354a, 767, 768, 780, 881). According to BERKELEY, arithmetic operations therefore cannot be introduced (on an ontological basis) by using sets of units as it is done, e.g., in EUCLID*'s 'Elements'*, but only (on a nominalistic basis) by means of the Indo-Arabic algorithms.

In his work *'Arithmetica absque Algebra aut Euclide demonstrata'* (in *'Works',* Vol. 4), BERKELEY only dealt with the algorithms of addition, subtraction, multiplication and division in the decimal place-value-system, the resolution of quadratic and cubic equations, etc. These can also be treated without problems within the framework of a nominalistic view. However, great difficulties arise if one wants to explain (on the narrow basis of BERKELEY*'s* arithmetic) what these algorithms should mean. *That "multiplication" is the same for* EUCLID *and* BERKELEY *can only be proved by using the general associative, commutative and distributive laws.* But these laws are not available in BERKELEY's arithmetic. In general, laws that are supposed to apply to *all* numbers (e.g., $\forall x \forall y$: $x + y = y + x$) cannot be proved in BERKELEY's arithmetic, since they cannot be derived from finitely many empirical observations. In particular, the principle of complete induction is not available in BERKELEY's arithmetic.

As long as arithmetic is only needed for the needs of everyday life (counting money while shopping, etc.), the *nominalistic view* favoured by BERKELEY may be sufficient, but scientific arithmetic cannot be performed on this basis. The series of natural numbers would only be available as a potentially infinite series, and many classical results would not be so easy - if at all - to prove. Number theory, in full generality, would no longer be possible, in particular, because of the unavailability of the principle of complete induction.

[1] For example, one can perceive *"one"* stone with the senses, but not *"unity"*. Likewise, one can perceive *"two"* animals, but no *"duality"*, etc.

BERKELEY flatly rejected Euclidean geometry, for points without extension and lines without breadth are not perceptible in the sensualist sense, and therefore do not exist (cf. BERKELEY's paper *'The Analyst'*, 1734, in *'Works',* vol. 4, op. cit.).

Following LUCRETIUS' *'De Rerum Natura'*, I, 749-752, BERKELEY sketched an alternative geometry in which *points* are extended patches that are just perceptible. His geometry is based on the concept of the smallest possible perceptible size (*"minimum sensible geometric magnitude"*). *Lines* are strokes whose breadth is no longer perceptible to the senses. He consequently rejects the infinite divisibility of lines and angles.

In BERKELEY's geometry, the side and diagonal of a square are commensurable, since both of them are composed of the *'smallest possible perceptible quantities'* in finite numbers (cf. *'Philosophical Commentaries',* No. 258). Therefore, squares cannot be doubled and the Pythagorean theorem is not valid in BERKELEY's geometry:

> *„One square cannot be double of another. Hence the Pythagoric theorem is false."*
> BERKELEY: '*Philosophical Commentaries',* no. 500.

BERKELEY also rejects the concept of congruence, because he believes that, if you place one drawn triangle on top of another, the triangle underneath is no longer visible, and therefore does not exist. He writes:

> *„The Mathematician should look to their axiom* Quae congruunt sunt aequalia. *I know not what they mean by bidding me put one Triangle on another, the under Triangle is no Triangle, nothing at all, if not being perceived."*
> BERKELEY: *'Philosophical Commentaries'*, Nr. 528.

All of this makes his alternative geometry less plausible. There are probably only negative results in it; there seem to be no positive statements.

He did not want to allow mathematics to deal with infinity, leaving that to theology alone. Infinite objects are not perceivable through the senses. So, for BERKELEY, even the basic concepts of the then-new infinitesimal mathematics proved to be nonsensical. For BERKELEY, an infinitely small size is nothing else but 'nothing'. He called people who feel the need to deal with such nothings "*nihilarians"*. The infinitesimal quantities are, for BERKELEY: *"Ghosts of departed Quantities"* (BERKELEY: '*The Analyst'*, § 35).

It follows from all these quotations that BERKELEY was not prepared to accept the view of mathematics as a *'framework of concepts'* (*'Fachwerk von Begriffen'*), as DAVID HILBERT put it, an idea that had prevailed since antiquity. If the terms do not describe sense-perceptible objects, then - this is how one would like to interpret BERKELEY - they may be good for fairy tales and ghost stories, but not for a discipline such as mathematics. BERKELEY's critical remarks of the mathematicians' approach had been justified in some cases (for example, in the case of infinitesimal calculus, which had not yet been thought through at the time with sufficient care) and helpful, but his strict empiricism was not suitable as a basis for the development of classical mathematical disciplines.

11.2 DAVID HUME's scepticism

DAVID HUME was a Scottish diplomat, historian and philosopher. He was born in Edinburgh on May 7, 1711, and died there on August 25, 1776. In 1739-1740, HUME

published his major work: '*A Treatise of Human Nature, Being an Attempt to introduce the experimental method of Reasoning into Moral Subjects*'. The book did not bring him the hoped-for success. He subsequently reworked it and published the new version under the title *'Essay'*.

In *'Treatise',* HUME discusses the concept of space in the first part of the first book. For HUME, the objects of geometry are the *'ideas'* that we have of the sizes of space. He uses the word somewhat differently from LOCKE or BERKELEY. For HUME, *'ideas'* are the *"faint images"* that we form in our soul of the *'vivid impressions'* that our senses get from the sizes of space.

He discusses in detail the question as to what a *mathematical point* is (loc. cit., page 88 ff.). He starts from the usual definition of mathematicians: A point (in 3-dimensional space) is a spatial quantity that has neither length, nor breadth, nor depth, and is therefore indivisible (loc. cit., page 96). According to HUME, such a definition only makes sense if there are indivisible spatial quantities (*'extensions'*). If the concept of a point were only present in the soul, and if points were not present anywhere in the sense-perceivable world, then the concept of a point (for HUME) would be unsatisfactory and empty. So, points must be spatial quantities in the sense-perceptible world. In his 'essay', he writes

„The idea of extension is entirely acquired from the senses of sight and feeling,"

and, a little later:

„an extension that is neither tangible nor visible, cannot possibly be conceived."

Finally, HUME refers to spatial quantities as 'mathematical points' if they appear indivisible to the sense of sight or the sense of touch. He means that such spatial quantities exist. As a 'proof', he writes that ink spots on a piece of paper, which can be seen from a sufficiently large distance, where they were still visible a moment ago, appear to the eye as indivisible objects (op. cit., page 95). For HUME, a limited line is always composed of finitely many indivisible sections (op. cit., p. 78). In section 11.2 on 'BERKELEY's Critique', we have already seen the conclusions to which such views lead.

11.3 JOHN STUART MILL's critique

The main representative of empiricism in the 19th century was the English philosopher, psychologist and sociologist JOHN STUART MILL (born on May 20, 1806, in London, died on May 8, 1873, in Avignon).

His views on epistemology were set out in detail in his influential book *'A System of Logic, ratiocinative and inductive'* (London, 1843). For him, the only source of knowledge is the experience conveyed by the senses, and the only permissible method for obtaining general knowledge is (incomplete!) *induction* in the sense of FRANCIS BACON (1561-1626). He believes that, in geometry, too, individual insights are gained through sensory perception and general knowledge is gained by inductive means (op. cit., page 147).

Concerning **geometry**:

MILL adopts from LUCRETIUS and BERKELEY the concept of what a 'point' is and writes (op. cit., page 148, column 1)

„the minimum visible, the smallest portion of surface we can see."

He accepts the definition of the 'point' as given by EUCLID, but thinks that it is nothing more than an allusion to the above concept of the sense-perceptible point.

He expresses himself similarly about the concept of 'straight lines' as 'length without breadth'. In MILL, we read (op. cit., page 148):

"We can reason ... , but not conceive a line without breadth."

He asserts that lines that we see with our physical eyes, or with our spiritual eyes, always have a certain breadth. He writes:

"If any one doubts this, we may refer him to his own experience."

On page 148, column 2, he finally summarises his views as follows:

„Since, then, neither in nature, nor in the human mind, do there exist any objects exactly corresponding to the definitions of geometry, while yet that science cannot be supposed to be conversant about non-entities; nothing remains but to consider geometry as conversant with such lines, angles, and figures as really exist; and the definitions, as they are called, must be regarded as some of our first and most obvious generalisations concerning those natural objects. The correctness of those generalisations, as generalisations, is without a flaw: the equality of all the radii of a circle is true of all circles, so far as it is true of any one: but it is not exactly true of any circle; it is only nearly true; so nearly that no error of any importance in practice will be incurred by feigning it to be exactly true."

Similar to ARISTOTLE, MILL also claims that there are no independent mathematical objects. Since geometry should not refer to something that does not exist, the objects of geometry are identified with the objects of the natural real world.

But there is an important difference between the views of ARISTOTLE and MILL, for MILL (unlike ARISTOTLE) is willing to pay the price of identification,

"... indem er die notwendige Gültigkeit der geometrischen Sätze auf die Geltungsebene empirischer Generalisierungen herabmoduliert - und damit seine Theorie der Mathematik hinreichend unplausibel macht." GÜNTHER PATZIG, op. cit., p. 120.

[... by modulating the necessary validity of geometric propositions down to the level of validity of empirical generalizations - and thus making his theory of mathematics sufficiently implausible.]

According to MILL, geometry would be a science that makes statements about facts that only exist approximately.

Concerning **Arithmetic:**

Arithmetic is called a "*stronghold of nominalism*" (page 167, column 1). He writes (op. cit., page 167, column 2) that numbers do not exist as abstract things:

"All numbers must be numbers of something; there are no such things as numbers in the abstract."

MILL thinks that one can only speak of *'ten apples'* or *'ten pulses',* for example, but not of an abstract *'ten'.* Mathematical propositions in which numerals occur are therefore not

statements about abstract numbers, but statements that deal with objects of the natural real world and are always true (p. 167, column 2). The equation *2 + 1 = 3*, for example, expresses an infinite number of *'physical facts'* (page 168, column 2) for MILL. He mentions the example:

> *"Two pebbles and one pebble are equal to three pebbles"* (page 168, column 2).

According to MILL, the validity of arithmetic propositions can only be shown by inductive means. Arithmetic assertions, such as *3 + 7 = 11*, can, however, be refuted by sensorial experience!

According to MILL, the number *2* is different from the number *3* because two pebbles are different from three pebbles.

> *„The fundamental truths of that science all rest on the evidence of sense; they are proved by showing to our eyes and our fingers that any given number of objects, ten balls, for example, may by separation and rearrangement exhibit to our senses all the different sets of numbers the sum of which is equal to ten."* (p. 169, column 1).

The natural numbers are introduced by MILL (through recursion) as words with which the powers of finite quantities of sense-perceivable objects can be expressed. The number 3, for example, is, for him, a word that serves to denote the number of elements of a set of sense-perceptible objects whenever this set can be divided into a two-element and a one-element subset, so that this division is also sense-perceivable (p. 169). MILL writes:

> *„Three is two and one, ... (because) collections of objects exist, which while they impress the senses thus* $^{0}_{0}{}^{0}$*, may be separated into two parts, thus* 00 0. *... We call all such parcels Threes."* (p. 169, column 1–2).

EDMUND HUSSERL could only shake his head over these definitions of the individual numbers 2, 3, 4, 5, ... and would have liked to ask MILL which sensory perceptions can be used, if one wants to speak of 'three' judgments or 'three' impossibilities, for example (cf. HUSSERL: *'Philosophie der Arithmetik'* (1891), op. cit., p. 17).

GOTTLOB FREGE (*'Die Grundlagen der Arithmetik'*, Breslau 1884, p. 9) wrote, somewhat mockingly:

> *"Er* [MILL] *belehrt uns nämlich, daß jene Definitionen* [d.h. die Definitionen der einzelnen Zahlen 1, 2, 3, 4, ...] *keine* [Definitionen] *im logischen Sinne seien, daß sie nicht nur die Bedeutung eines Ausdrucks festsetzen, sondern damit auch eine beobachtete Thatsache behaupten. Was in aller Welt mag die beobachtete oder, wie Mill auch sagt, physikalische Thatsache sein, die in der Definition der Zahl 777864 behauptet wird? Von dem ganzen Reichthume an physikalischen Thatsachen der sich hier vor uns auftut, nennt uns Mill nur eine einzige, die in der Definition der Zahl 3 behauptet werden soll. Sie besteht nach ihm darin, daß es Zusammenfügungen von Gegenständen gibt, welche während sie diesen Eindruck* $^{0}_{0}{}^{0}$ *auf die Sinne machen, in zwei Theile getrennt werden können, wie folgt: 00 0. Wie gut doch, daß nicht alles in der Welt niet- und nagelfest ist; dann könnten wir diese Trennung nicht vornehmen, und 2+1 wäre nicht 3. Wie schade, daß Mill nicht auch die physikalischen Thatsachen abgebildet hat, welche die Zahlen 0 und 1 zu Grunde liegen."*

> [For he, MILL, teaches us that those definitions (i.e. the definitions of the individual natural

numbers *1, 2, 3, 4, ...*) are not definitions in the logical sense, that they not only establish the meaning of an expression, but also assert an observed fact. What in the world might be the observed or, as Mill also says, physical factuality claimed in the definition of the number *777864*? Of the whole wealth of physical facts that is unfolding before us here, Mill names only one that is to be asserted in the definition of the number *3*. It consists, according to him, in the fact that there are combinations of objects which, while giving this impression of 0_0^0 to the senses, can be separated into two parts, as follows: 00 0. How good it is that not everything in the world is nailed down; then we could not make this separation, and *2+1* would not be *3.* What a pity that Mill did not also depict the physical facts on which the numbers *0* and *1* are based!]

FREGE also criticizes MILL for referring only to the 'impressions of the senses' of the number *'three'*, for example, in the form 0_0^0 or 00 0, and thinks that it would then be wrong to speak of *'three'* pulse beats, or of the *'three'* taste sensations *sweet, sour, bitter*, or of the *'three'* zeros of a cubic equation (FREGE, loc. cit., pp. 9-10).

Because MILL's equations of elementary arithmetic are statements about the numbers of objects in the natural real world, they can only be obtained inductively, and therefore cannot claim (according to MILL, page 169, column 2) to be *'exact, universally valid truths'*. It is remarkable that MILL admits this consequence of his approach to arithmetic. But this does not make his arithmetic more attractive.

According to MILL, arithmetical judgements are

„fundamental truths, ... that rest on the evidence of sense",

and, hence, are empirical judgments. But we have already seen, in Chapter 9, when we discussed the views of JOHN LOCKE, that this is not true. We explained, using the concrete example *4 + 3 = 7*, that this equation is true because the rules of arithmetic calculus were correctly applied, and that it is not possible to check with the senses, but only with the mind, whether the rules were *correctly* applied.

These arguments show that elementary arithmetical judgments are not empirical judgments.

R.L. GOODSTEIN (op. cit.) makes the apt comparison: if, in a game of chess, the king has come into check, then the situation does not exist on the basis of an empirically (i.e., physically) ascertainable movement of a chess piece, but on the basis of the rules of the game of chess.

11.4 Discussion

With the discovery of incommensurable quantities by the Pythagoreans, a geometry was created in antiquity in which points have no extension, lines have no breadth and surfaces have no thickness. These are objects that are not perceptible to the senses and exist only for thinking (cf. Chapter 1). With this discovery, something new was created, something that the Greeks called 'mathematics'. The ontological and epistemological problems raised by this new discipline were worked on by many philosophers and mathematicians from PLATO onwards, and they gained many deep insights, - but they were also, often enough, mistaken.

But the philosophers from the camp of the empiricists have particularly disappointed us and left us quite perplexed: they do not accept anything of what has been known as mathematics since the Greeks.

Neither BERKELEY, nor HUME, nor MILL were able to achieve their goal of constructing mathematics on an empirical basis. They could not achieve the goal of explaining the basic concepts in such a way that it would be possible to establish general laws and general theorems that apply to *all* (!) geometric objects of a certain genus, or to *all* (!) numbers, etc.

But they were nevertheless right in their critical remarks pointing out certain shortcomings in mathematics, and they also made various useful remarks about the intended meanings of the basic concepts. Some of these remarks were taken up in later times. For example, the geometrician MORITZ PASCH (1843-1930) wrote, in his famous *'Lectures on recent developments in geometry'* (*'Vorlesungen über neuere Geometrie',* op. cit., p. 3):

> *"Allemal aber werden die Körper, deren Teilung sich mit den Beobachtungsgrenzen nicht vertägt, »Punkte« genannt."*
>
> [But always the bodies whose division is incompatible with the limits of observation are called »points«".]

This is a reference to DAVID HUME. But, unlike HUME, PASCH *does not* understand this statement *as a definition* of what 'points' are, but only as a statement of the *intuitive content* of the concept of a 'point'. PASCH divides his presentation of geometry into two carefully separated parts (following TSCHIRNHAUS and WOLFF, cf. Chapter 10, sction 10.9): in the first part, the prolegomena, the intuitive contents of the various basic concepts are discussed, and in the second part, the main part, the idealizations and formalizations that lead to the actual foundations of geometry are worked out, namely, as a system of axioms in the form of implicit definitions. The first part, therefore, deals with the contentual aspects of *descriptive geometry* and the second part with the setting up of axioms for a *theoretical geometry*. The axioms should formalise the previously established contents as precisely and comprehensively as possible. In this way, on the one hand, an anchoring of the theoretical geometry in reality is achieved and, on the other hand, a mathematical theory is constructed that allows for universally valid statements to be strictly proven.

The empiricism advocated by BERKELEY, HUME, MILL and others can contribute to the prolegomena of mathematical theories, but nothing to the actual formation of the geometrical theory itself.

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Chapter 12
KANT's Conception of Mathematics

The Age of Enlightenment produced two great philosophical schools of thought, *'rationalism'* (RENÉ DESCARTES, GOTTFRIED WILHELM LEIBNIZ and others) and *'empiricism'* (GEORGE BERKELEY, DAVID HUME and others). The fundamental difference between these two schools of thought *'concerned the measurement of the extent of our knowledge a priori, that is, the knowledge we can have independently of our sense-perception. The rationalists had the tendency to regard this scope as large, the empiricists, however, as small'* (quoted after ERHARD SCHEIBE, op. cit., p. 355). IMMANUEL KANT set himself the goal of redefining this scope.

12.1 KANT's curriculum vitae

IMMANUEL KANT was born on April 22, 1724, in Königsberg (in Prussia). At the age of 16, he began his studies of mathematics, natural sciences and philosophy at the University of Königsberg, completing them six years later. Afterwards, he worked as a private teacher. In 1756, he became a 'Privatdozent' and, in 1770, at the age of 46, he became 'Professor of metaphysics and logic' at the University of Königsberg.

He gave his lectures at seven o'clock in the morning. KANT's student REINHOLD BERNHARD JACHMANN (op. cit., p. 29) described KANT's lecture style:

> *"Eine besondere Kunst bewies Kant bei der Aufstellung und Definition metaphysischer Begriffe dadurch, daß er vor seinen Zuhörern gleichsam Versuche anstellte, als wenn er selbst anfinge, über den Gegenstand nachzudenken, allmählig neue, bestimmende Begriffe hinzufügte, schon versuchte Erklärungen nach und nach verbesserte, endlich zum völligen Abschluß des vollkommen erschöpften und von allen Seiten beleuchteten Begriffes überging und so den streng aufmerksamen Zuhörer nicht allein mit dem Gegenstande bekannt machte, sondern ihn auch zum methodischen Denken anleitete."*
>
> [KANT evinced a special art in the setting up and defining of metaphysical concepts by making experiments in front of his listeners as if he himself began to think about the object, gradually adding new defining concepts, already attempting to improve explanations bit by bit, finally moving on to the complete conclusion of the completely exhausted and from all sides illuminated concept, and thus not only acquainting the strict attentive listener with the object, but also guiding him to methodical thinking.]

KANT must have been a spirited speaker, not only in his lectures, but also in private

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U. Felgner, *Philosophy of Mathematics in Antiquity and in Modern Times*,
Science Networks. Historical Studies 62, https://doi.org/10.1007/978-3-031-27304-9_12

circles. He enjoyed having guests around him. He invited them to lunch. The meal was frugal, but everyone found a small bottle of red Medoc/Bordeaux next to their plate. It was required that the number of guests be no less than 3 and no more than 9. He justified this by saying that 3 was the number of the Graces and 9 the number of the Muses (cf. JACHMANN, op. cit., pp. 139, 146, 169). KANT's life was otherwise highly disciplined and uniform.

In 1781 (at the age of 57), KANT published his most important work, the *'Kritik der reinen Vernunft'* (*Critique of Pure Reason).* For the second edition of this *'Kritik'*, published in 1787, he rewrote some parts. One therefore distinguishes these two versions with the letters A (1st edition) and B (2nd edition). We use KrV as an abbreviation for the title *'***K***ritik der* **r***einen* **V***ernunft'.*

In 1788. KANT published a *'Kritik der praktischen Vernunft' (Critique of Practical Reason)* and, in 1790, a *'Kritik der Urteilskraft' (Critique of Judgment).* - He gave his last lecture in 1796.

KANT died in 1804 on February 12. The funeral procession was followed by almost the entire population of Königsberg, as well as people from all over Germany.

12.2 KANT's *'Critique of Pure Reason'*

Soon after the publication of *'Kritik der reinen Vernunft'*, it became one of the most important books in the field of philosophy. M^me^ DE STAËL-HOLSTEIN wrote, in 1810, in her book *'De L'Allemagne'* (Paris-London 1810/1813, Volume 3, p. 67-68):

> *"... mais lorsqu'enfin on découvrit les trésors d'idées qu'il renferme, il produisit une telle sensation en Allemagne, que presque tout ce qui s'est fait depuis lors, en littérature comme philosophie, vient de l'impulsion donnée par cet ouvrage."*
>
> [... but when the treasures of ideas contained therein was finally discovered, it caused such a sensation in Germany that everything that has come to light since then in the fields of literature and philosophy has been due to the impetus given by that work.]

However, the book has also provoked contradiction, and its title has confused some readers. JOHANN GOTTFRIED HERDER (1744-1803), for example, wrote [1]:

> *"(...) der Titel befremdet. Ein Vermögen der menschlichen Natur kritisiert man nicht; sondern man untersucht, bestimmt, begränzet es, zeigt seinen Gebrauch und Mißbrauch."*
>
> [... the title is strange. One does not criticize a faculty of human nature; one examines, determines, limits it, shows its use and abuse.]

HERDER has thoroughly misunderstood the word *'criticism'* here. HERDER understands the word *'to criticize'* only colloquially, in the modern sense of *'to blame, to disapprove'*. He probably had not thought of the Greek root of the word, for, here, *'to criticize'* actually means 'to determine, to limit, to judge the use and abuse'. The word *'criticism'* is derived from the Greek *'krinein'* (κρίνειν). This verb means, first of all:

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[1] See J.G. HERDER: *'Verstand und Erfahrung, Vernunft und Sprache - eine Metakritik zur Kritik der reinen Vernunft'*, 1799, in HERDER*'s 'Sämmtliche Werke'*, section: *'Zur Philosophie und Geschichte',* volume 14, Karlsruhe 1820, page 1.

(1) *to discriminate (between),* and, in particular, *to decide, to determine*,
(2) but also: *to judge, to sit in judgment, to (judicially) decide*,
(3) and even: *to accuse, to indict, to convict*.

Thus, the original meaning of the Greek word *'to criticise'* (κρίνειν) does not mean *'to find faults'* and the word *'criticism'* does not mean *'fault finding'*.

However, in order to understand the Greek word *'krinein'* (κρίνειν) a little better, it should be mentioned that it has several derivations that are perhaps better known, namely, the word *'criterion'* (κριτήριον), which is *'the decisive mark of distinction'*. Also, *'kriter'* (κριτήρ) is a deciding judge and *'krites'* (κριτής) is a deciding umpire or referee in a dispute or in a Cricket match, etc. In particular, a *'crisis'* (κρίσις), in the original meaning of the word, is not necessarily a time of difficulty, danger or anxiety about the future, as it is usually understood nowadays, but just a state in which a definitive decision must be made.

KANT's *'Kritik'*, first of all, wants to determine and distinguish the different cognitive faculties of *sensuality, understanding and reason*. But KANT's *'Kritik'* also wants to 'judge' the abilities of pure reason and to 'condemn' claims to abilities that are set too high. As an example, KANT's criticism of the ontological proof of the existence of God, which goes back to ANSELM of Canterbury and RENÉ DESCARTES, needs to be mentioned.

KANT worked on his book for about ten years. In a letter to MARKUS HERZ dated February 21, 1772, KANT wrote that he was working on a book that could be entitled *'Die Grenzen der Sinnlichkeit und der Vernunft'* (*The Limits of Sensuality and Reason*). The determination of these limits is the main subject of the *'Kritik der reinen Vernunft'*.

KANT's *'Kritik'* refers to *'reine Vernunft'* (*pure reason*). The German word *'Vernunft'* is derived from the verb *'vernehmen'*, a verb that has two slightly different meanings. Here, the intended meaning is in the sense of *'to interrogate, to find out by inquiry, to examine'*, i.e., *'vernehmen'* in the sense of *'auseinandernehmen, ausfragen, befragen, verhören'*. Hence, a good translation of the term *'Vernunft'* into Latin is *'intellectus'* (cf. Epilogue, p. 296). The term *'ratio'* (*'reason'* in English) does not mean quite the same as *'Vernunft'*.

'Pure reason' (according to KANT) is the activity of *'finding out by inquiry'*, i.e. *reasoning,* which does not depend on sense-perception (cf. KrV: B24, and also JAKOB FRIEDRICH FRIES, *'System der Logik'* (1819), op. cit., p. 92).

The title of KANT's main work, *'Critique of Pure Reason'*, has thus become somewhat clearer. In this work, KANT intends to critically examine the main philosophical currents of his time, *rationalism* (DESCARTES, LEIBNIZ, WOLFF et al.) and *empiricism* (DAVID HUME et al.), and, in doing so, to clarify the true capabilities of reason. The title *'Critique of Pure Reason'* suggests that KANT claims to have followed his great predecessors:

- RENE DESCARTES: *'Meditationes de Prima Philosophia'*, Paris 1641.
- JOHN LOCKE: *'An Essay concerning humane understanding'*, London 1690,
- GOTTFRIED WILHELM LEIBNIZ: *'Nouveaux Essais sur l'entendement humain'* (written around 1704, published posthumously by R.E. RASPE in Amsterdam and Leipzig 1765),
- ETIENNE BONNOT DE CONDILLAC: *'Essai sur l'origine des connoissances humaines'*, Amsterdam 1746, and

- DAVID HUME: *'Philosophical Essays concerning human understanding'.* London 1748,

and has advanced to a final answer to the question of the possibilities and limits of human cognitive faculty (reason). In doing so, he relies on the distinctions:

a priori - a posteriori and *analytic - synthetic.*

Of great importance is the contrasting of

intuition (Anschauung) versus *concept (Begriff),*

and of even greater importance the distinction between *'empirical intuition'* ('sinnliche Anschauung') and *'pure intuition'* ('reine Anschauung'). - Let us first explain the first two pairs of terms mentioned above.

12.3 The distinction: *a priori - a posteriori*

In Latin, *a priori* means 'earlier in time' or 'prior to... '. In logic, a judgment ('Urteil') is called '*judgment a priori*' when it is valid 'from the beginning', i.e., without any further examination. In KANT's KrV, however, a judgment is called '*judgment a priori*' if it is valid without any further empirical examination.

In Latin, *a posteriori* means 'later in time'. In KANT, a judgment is called *'judgment a posteriori'* if it is based on empirical experience (i.e., on perception through the senses).

According to KANT, an assertion is a *'judgment'* if it is a *true* assertion. This is how the word is explained in the etymological dictionaries: a 'judgment' is a *'verdict given by a judge*' and a 'verdict' is a 'dictum of a verity'.

KANT also speaks of *'knowledge a priori*' if it concerns knowledge that can be attained 'before' all experience, that is, if it is independent of all sense-perceptions and can only be gained through mental reflection.

KANT speaks of *'knowledge a posteriori*' if it concerns knowledge that is only possible 'after' experience, i.e., with the help of experience (i.e., on the basis of sense-perceptions). Thus, *'knowledge a posteriori*' is based on empirical sources (KrV: Introduction, B2).

KANT is convinced that all human knowledge has two sources, namely, *sensuality and understanding* ("*Sinnlichkeit und Verstand"*), (KrV: B29). It follows from this that judgments are either *a posteriori* or *a priori.*

Are there judgments a priori? KANT thinks that all theorems of pure mathematics are judgments a priori (KrV: B4, B8, B14). Thus, he does not share the opinion of the empiricists.

But neither does KANT share the opinion of the rationalists, who wanted to draw the propositions of pure mathematics entirely from the ratio (the intellect), i.e., as *analytic judgments a priori*. KANT does not regard them as analytic judgements, but believes that all of them are *synthetic judgments a priori.*

12.4 The distinction: *analytic - synthetic*

KANT discusses the distinction between *analytic* and *synthetic* judgments in the

introduction to the *'Critique of Pure Reason',* B10-B14. He calls a judgment *analytical* when nothing needs to be added to its verification, hence, when it can be recognized as true solely through dissection (analysis) of the concepts and terms that appear in the judgment. We can also put it this way: *A judgment is an 'analytic judgment' if its truth results solely from the knowledge of the definitions of all concepts and all terms occurring in the judgment.* A judgment is therefore an 'analytic judgement' if the cause of its truth is a 'formal cause' (*causa formalis*) in the sense of ARISTOTLE - see Chapter 3 and Chapter 7.

The term 'analytic' (or 'analytical') chosen by KANT is based on the classical use of this word in logic. Statements that are 'logically certain' were called *'analytical'* by ARISTOTLE (ἀναλυτικός); LEIBNIZ called them *'identical judgments'* - see Chapter 10, footnote 4.

KANT also occasionally formulated the concept of analyticity as follows (in reference to HOBBES):

> A judgment is *analytic* if the concept of the predicate is contained in the concept of the subject (KrV: B10, but see also B. BOLZANO *'Wissenschaftslehre'*, 1837, vol.2, §148, p. 88),

and also (in reference to LEIBNIZ):

> A judgment is *analytic* if it can be proved from the 'principle of contradiction' ("Satz vom Widerspruch") alone (KrV: B 190-193; and *'Prolegomena'* § 2, b).

The concept of analyticity can be expressed, more carefully formulated, as follows:

Definition. A statement Φ is an *analytic judgment* if it is possible to transform Φ into a (logically) universally-valid statement (i.e., a true *'identical sentence'* - see footnote 4 in Chapter 10) by replacing some concepts or terms appearing in Φ with their definitions.

Using FREGE's judgment dash ⊢, the concept of analyticity can also be defined as follows:

Definition (alluding to HOBBES, LEIBNIZ, TSCHIRNHAUS, WOLFF et al., see Chapter 10). A statement Φ is an *analytical judgment* if its truth can be deduced in a purely logical way from the definitions of all concepts and terms B_1, B_2, ..., B_n appearing in Φ, i.e., if it is the case that the truth of Φ can be proven on the basis of all definitions of the concepts and terms appearing in Φ:

$$\{Def(B_1), Def(B_2), \ldots, Def(B_n)\} \vdash \Phi.$$

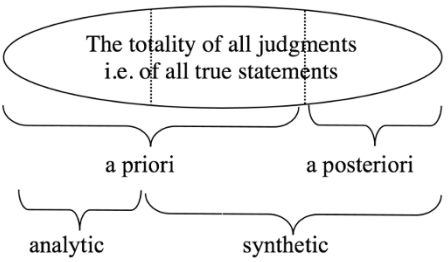
Here, Def(*B*) denotes a definition of the concept or term *B*, i.e., a set of records from which everything that belongs to the definition of *B* can be taken.

Notice that a judgment Φ is an *analytic judgment* if it can be proved from the system of all definitions of the concepts and terms appearing in Φ *without presupposing any postulates or axioms* - unless one knows that the postulates (or axioms) are all also analytical judgments and, hence, are 'true' as judgments!

KANT calls a judgment *'synthetic'* when it makes a statement that cannot be derived solely from the logical-conceptual analysis of the concepts and terms occurring in it. If one

wants to confirm a synthetic judgment, a contribution is necessary that complements the logical-conceptual analysis. This contribution can come from sense-perception, but may also possibly be drawn from completely different sources (KrV: A6-7; B10). Most of the judgments that KANT discusses are synthetic, since, for their confirmation, a *synthesis of concept and intuition* must be carried out.

All analytic judgments are obviously a priori and all a posteriori judgments are synthetic. We describe these relationships in the following diagram.



KANT posed the profound question of whether there are judgments that are *synthetic* as well as *a priori*. If they do exist, then they are true statements that must be based on *intuition* (in a broad sense) in their proofs, but, at the same time, are independent of *empirical intuitions*. - How is that possible? - So, KANT*'s* question is:

> *"Wie sind synthetische Urteile a priori möglich?"* (KANT, KrV: B19).
> [How are synthetic judgments a priori possible?]

This is the central question in KANT*'s 'Critique of Pure Reason'*. It is a question that concerns not only the capacity of *understanding*, but also the *empirical*, as well as the *non-empirical*, *intuition*. It is a question that (according to KANT) transcends the sphere of *understanding* and concerns the capacity of *'pure reason'*. Therefore, KANT*'s* work is neither an *'Essai sur l'entendement humain'* nor an *'Essay concerning human understanding'*, as in LOCKE, LEIBNIZ, HUME and CONDILLAC, but rather a work on *'pure reason'*. - With this, the title of KANT's work has finally become a bit clearer to us.

The answer to the question mentioned above was given by KANT himself. He thought *that all propositions of mathematics were synthetic judgments a priori*, because only a certain portion of empirical intuition is relevant in mathematics, namely, so-called *"pure intuition"* (*'reine Anschauung'*). He was convinced that a judgment that contains only concepts that can be represented in pure intuition is always a synthetic judgment a priori.

In the following, we want to report on this in detail. We begin with KANT's claim that all theorems of (Euclidean) geometry and all theorems of (elementary) arithmetic are synthetic judgments. From this, KANT draws the conclusion that the rationalists were wrong when they claimed that all arithmetic and all geometric propositions could be drawn solely from definitions, and therefore were to be regarded as analytic judgments (cf. Ch. 7 and Ch. 10).

12.5 The synthetic character of geometric propositions

In the 5th section of the introduction to the *'Critique of Pure Reason'* (KrV: B14), KANT makes the bold claim: *'Mathematische Urteile sind insgesamt synthetisch'* (*Mathematical judgments, on the whole, are synthetic*). To substantiate this thesis, he discusses one example each from (Euclidean) geometry and from (elementary) arithmetic. In geometry, he considers the classical *Triangle-Sum-Theorem* (EUCLID, *'Elements'*, I.32), which states that, in every triangle, the sum of the interior angles is equal to two right angles. We have already discussed this theorem in Chapters 7 (sections 7.3 & 7.4) and 8 (sections 8.4, 8.5 & 8.7). In the theorem, a property of all triangles is claimed. KANT asserts that this property does not belong to the definition of a triangle and that the theorem, therefore, is not at all analytic.

We recall that, by definition, triangles consist of three non-collinear points (the vertices of the triangle) and their three connecting lines.

The proof of the *Triangle-Sum-Theorem* is as follows: Consider a triangle with the vertices *A, B, C* and the sides *a, b, c*. One needs another straight line, namely, one that passes through one of the three vertices, is parallel to the opposite side of the triangle and is uniquely determined. Since such a parallel is not present, when a triangle is given, it must be obtained somehow. Its existence cannot be deduced from the definition of a triangle. It must be constructed in intuition.[2] As soon as it is available, the (well-known) proof can be continued as indicated in the *'Elements' of* EUCLID, Book I, § 32. Since the proof makes use of the construction of a parallel straight line, which is carried out in intuition, the theorem proved in this way is a synthetic judgment, Q.E.D.

Notice that KANT's argument concerns classical geometry as it is developed in the *'Elements'* of EUCLID, Book I, § 32. This is a geometry in which mathematical objects are available when they are (mentally) constructed. At the end of the 19th century, this understanding of geometry changed considerably when the *structuralist* point of view became common among mathematicians (cf. Chap. 19 and Chap. 20). In fact, in a geometric structure, which is a model of all Euclidean postulates (or axioms), all geometric objects are supposed to be there right from the beginning. This, however, is not the case in classical geometry, and it is quite important to observe this difference!

12.6 The synthetic character of arithmetic theorems

In order to show that, in arithmetic, all judgments are synthetic, KANT discusses the simple proposition *7 + 5 = 12*. KANT has chosen this numerical example somewhat playfully, because it contains a hidden allusion to the Platonic *'Theaitetos'*. In fact, the Platonic *'Theaitetos'* is about the question: *What is knowledge and what is cognition?*, and it is discussed there (195e-200a) whether it is impossible to believe that *5 + 7* is *11*, and, if so, why it should be impossible. KANT writes:

[2] This is correct, because, in non-Euclidean geometries, there are no such uniquely defined parallels and, here, the sum of angles is not equal to two right angles. KANT, by the way, contradicts an assertion by ARISTOTLE, from the *'Second Analytics'* (73b32-74a), that the concept of a triangle would contain the statement that the sum of the inner angles is equal to two right angles.

> *"Man sollte anfänglich zwar denken, daß der Satz 7 + 5 = 12 ein bloß analytischer Satz sei, der aus dem Begriffe einer Summe von Sieben und Fünf nach dem Satz des Widerspruchs erfolge, ..."*
>
> [One should initially think that the proposition *7 + 5 = 12* is a merely analytic judgment, which is derived from the concept of a sum of seven and five according to the laws of logic],

but KANT considers the proposition *7 + 5 = 12* to be synthetic, and argues as follows:

> *"Der Begriff von Zwölf keineswegs schon dadurch gedacht ist, daß ich mir bloß jene Vereinigung von Sieben und Fünf denke, und, ich mag meinen Begriff von einer solchen möglichen Summe noch solange zergliedern, so werde ich doch darin die Zwölf nicht antreffen. Man muß über diese Begriffe hinausgehen, indem man die Anschauung zu Hilfe nimmt, die einem von beiden korrespondiert, etwa seine fünf Finger, oder (wie Segner in seiner Arithmetik) fünf Punkte, und so nach und nach die Einheiten der in der Anschauung gegebenen Fünf zu dem Begriffe der Sieben hinzutun. Denn ich nehme zuerst die Zahl 7, und, indem ich für den Begriff der 5 die Finger meiner Hand als Anschauung zu Hilfe nehme, so tue ich die Einheiten, die ich vorher zusammennahm, um die Zahl 5 auszumachen, nun an jenem meinem Bilde nach und nach zur Zahl 7, und sehe so die Zahl 12 entspringen. Daß 5 zu 7 hinzugetan werden sollen, habe ich zwar in dem Begriffe einer Summe = 7+5 gedacht, aber nicht, daß diese Summe der Zahl 12 gleich ist."* (KrV, Introduction, B15-16).
>
> [The concept of twelve is by no means already conceived by thinking only of a union of seven and five, and, while I may analyse my concept of such a possible sum, I will not find the twelve in it. One must go beyond these concepts by taking the intuition that corresponds to one of them, such as ones own five fingers, or (as SEGNER does in his Arithmetic) five points, and thus gradually add the units of the five given in intuition to the concept of seven. For I first take the number *7*, and, using the fingers of my hand for the concept of *5* as an illustration, I add the units which I had previously put together to make up the number *5*, and now, in my own image, I gradually add them to the number *7*, and thus see the number *12* emerge. That *5* should be added to *7,* I have thought in terms of a sum = *7 + 5*, but not that this sum is equal to the number *12*.]

We want to examine KANT's arguments and, for this purpose, we must have in mind the definitions of all terms and concepts that appear in the judgment "*7 + 5 = 12*". How were these terms and concepts defined at the time of KANT?

KANT was in possession of textbooks[3] by MICHAEL STIFEL, CHRISTIAN AUGUST HAUSEN, and ABRAHAM GOTTHELF KÄSTNER, and especially one by CHRISTIAN WOLFF: *'Die Anfangsgründe aller Mathematischen Wissenschaften, erster Theil', 1750*.

In these textbooks (as in PLATO and EUCLID, see Chapters 2 and 4), *'numbers'* are finite sets of units, i.e., *cardinal numbers*. According to WOLFF (op. cit., p. 39), the *'essence of a number'* consists in the fact *that one takes a number of units together* (*"daß man einerley Einheiten etliche mahl zusammen nimmt".*) It is tacitly assumed that the process of counting always leads to the same number, regardless of the order in which the objects are counted.

Furthermore: *Adding up means finding a number that is equal to different numbers taken*

[3] See the list in ARTHUR WARDA: *'Immanuel Kants Bücher'*, op. cit.

together (*"Addieren heißet eine Zahl finden, welche verschiedenen Zahlen zusammengenommen gleich ist"*, WOLFF, op.cit., p. 41). Here, addition is neither a 2-place function nor the formation of the union of disjoint sets of units, but rather an operation of counting, the execution of which takes a certain time.

From such descriptions, KANT deduces that *adding* two numbers thus means that

> *"... in tempore datu successive unum uni addendo."*
> [... in due course, one adds to some unit a further one.]

This is how KANT had already described it in his dissertation *'De mundi sensibilis atque intelligibilis forma et principiis',* 1770, Section III, § 15E (cf. also KrV: A163).

In the above definitions of the numbers *5* and *7* and of the concept of *addition,* as KANT rightly observes*,* it is not yet included that the process of adding five units to the number *7* eventually leads to the number *12*. The process must be carried out in a concrete way. It makes use of the surrounding space and the time available. The surrounding space is needed (according to KANT) to represent the number *5* concretely (in intuition), for example, through the five fingers of a hand, and the time to carry out the process of adding the five units step by step: *7 + 1 = 8*, *8 + 1 = 9*, etc. The judgment "*7 + 5 = 12*" does not result (according to KANT) from a logical analysis of the terms that appear, but only by using a representation of the Addendus[4] in intuition, and is, hence, a *synthetic judgment* (KrV: B15-16 and B205).

Today, we are used to introducing numbers and their addition in a completely different way. Thus, today KANT's line of reasoning is no longer convincing. But, on the basis of the definitions of the concept of the number and the arithmetical operations that were common in the 16th and 17th centuries (as well as long before and for long after), KANT could not come to any other conclusion. In his time, he drew entirely correct conclusions from the definitions available.

Nevertheless, mathematicians were by no means satisfied with KANT's argumentation, particularly because it was claimed that, in arithmetic, it was necessary to include space and time. But the mathematicians had to admit, at the same time, that they too had never given a perfect construction of the number system. KANT's handling of the concept of 'addition' was amateurish, but neither EUCLID (*'Elements'),* nor MICHAEL STIFEL (*'Arithmetica integra',* 1544), nor LEONHARD EULER (*'Algebra',* 1770), or anyone else had made a serious attempt to introduce the concept of the number in such a way that addition and multiplication, together with the laws upon which they are based, could be worked out comprehensively and logically, free from objection.

Today, we are in possession of a foundation of the concept of number and of arithmetic that is free from objection. After preliminary work by HERMANN GRASSMANN (1809-1877) in 1860, RICHARD DEDEKIND (1831-1916) first succeeded in establishing such a foundation in 1888, in his booklet *'Was sind und was soll die Zahlen*? (cf. Chapter 19). Let us now indicate how, on the basis of DEDEKIND's foundational work of arithmetic, KANT's question of the status of the equation *7 + 5 = 12* can be dealt with.

---

[4] In the sum *a* + *b* of two numbers *a* and *b, a* is the *augendus* (i.e., the number to be increased) and *b is* the *addendus* (i.e., the number to be added).

The natural numbers are introduced by DEDEKIND and GRASSMANN starting from the name of the number *1* by forming step by step names for their successors. Let ν(*m*) be the name of the successor of *m*, i.e., the number following *m.* Then, we have, as usual:

2 is the name of the number following *1*, so *2* = ν(*1*),
3 is the name of the number following *2*, so *3* = ν(*2*) = ν(ν(*1*)),
…
12 is the name of the number following *11*, so *12* = ν(*11*) = ν(ν(*10*)) =...
... etc. ….

Based on the successor function ν, the *addition* "+" *of ordinal numbers* (following GRASSMANN: *'Lehrbuch der Mathematik für höhere Lehranstalten',* Stettin 1860) can be defined in a recursive way as follows.

(*) $n + 1 = \nu(n)$,
(**) $n + \nu(m) = \nu(n + m)$.

Here, *n* and *m* are variables that run through the series of natural numbers. From (*) & (**), it follows that, for each augendus *n*, all sums $n + 1$, $n + 2$, $n + 3$, ..., $n + m$, $n + \nu(m)$, ... are explained. From (*) & (**), one can also derive the equation $7 + 5 = 12$ very easily:

$$7+5=\nu^6(1)+\nu^4(1)=_{(**)}\nu\left(\nu^6(1)+\nu^3(1)\right)=_{(**)}\nu\left(\nu\left(\nu^6(1)+\nu^2(1)\right)\right)=$$
$$=_{(**)}\nu\left(\nu\left(\nu\left(\nu^6(1)+\nu(1)\right)\right)\right)=_{(**)}\nu\left(\nu\left(\nu\left(\nu\left(\nu^6(1)+1\right)\right)\right)\right)=$$
$$=_{(*)}\nu\left(\nu\left(\nu\left(\nu\left(\nu^7(1)\right)\right)\right)\right)=\nu^{11}(1)=12.$$

From the definitions of the (names for the) numbers *2, 3, 4, 5, ..., 12* and the recursive definition of the addition '+', the identities of *7 + 5* and *12* can therefore be deduced logically without any problems. *But this proof does not provide an answer to the question as to whether 7 + 5 = 12 is an analytic or a synthetic judgment. The* reasons for this are as follows.

The identity *7 + 5 = 12* is - as we have just seen - *provable* in DEDEKIND's system of arithmetic, for instance, in the form of the DEDEKIND-PEANO axiomatic of the first order, DPA (in language with + and -). The proof is based on definitions and on axioms. The definitions of *5, 7* and *12* are: *5* = $\nu^4(1)$, *7* = $\nu^6(1)$ and *12* = $\nu^{11}(1)$, exactly the same as in KANT. Without using the axioms (*) & (**), 7 + *5* = 12 cannot be derived *from* the three definitions mentioned above. But, since these axioms are by no means (logically) universally-valid statements, one cannot conclude that *7 + 5 = 12* is an analytic statement. Neither can one conclude that *7 + 5 = 12* is a synthetic statement, for nowhere in the proof has intuition been used as an aid. One cannot even infer from the evidence that *7 + 5 = 12* is a *judgment* at all, i.e., a statement that is 'true' per se. DPA is merely a formal theory, in which the only question is whether something is provable or unprovable. It cannot be decided in a formal theory whether something is 'true', because 'truth' is a semantic concept. On the basis of the formal axiom system DPA, it cannot therefore be claimed that *7 + 5 = 12* is a *judgment* in the sense of KANT. We see that KANT*'s* questions, whether the

mathematical propositions are analytical or synthetic judgments, can only be answered in a historical context.

In the following, we return to a treatment of KANT's views in their historical context.

12.7 Of *'pure intuition'* and *'empirical intuition'*

The discussion of the examples from geometry and arithmetic, which were intended to prove the synthetic character of the theorems from these fields (see above), led KANT to the general conviction that, in mathematical proofs, not only must the definitions of terms and concepts be used, but also additional auxiliary quantities, which are to be constructed in intuition. These auxiliary quantities are not yet available at the beginning of a proof; they must be introduced during the execution of the proof. We introduce them by constructing them in our intuition. The introduction of the auxiliary quantities constructed in this way has the consequence that the proven statements are to be classified as synthetic statements. But, for these proofs to lead to synthetic judgments *a priori*, the construction must not take place in the empirical intuition; the construction must be free of any empirical admixture, i.e., it must be carried out in *'pure intuition'*.

For KANT, *"Anschauung"* ('Intuition') is basically *sensation*, i.e., perception through the senses. But sense-perception always includes a contribution from our own mind (our soul or our spirit), namely, the *'pure form'* of the object in front of us, as KANT says, KrV:B1. We add this pure form to the registered sensations. This process of adding the pure forms to the mere sensations is called *'pure intuition'* (intuitus purus, *'reine Anschauung'*) by KANT [5](KrV: B34-35).

A *straight line* drawn with a ruler on a piece of paper is - when viewed with a magnifying glass - an elongated pile of color spots that are not evenly distributed everywhere. This is what the eyes perceive as sensations. But the mind orders them, judges them and adds the pure form of the straight line to the sensual perception, perhaps even as a 'breadthless' line.

KANT describes the registration of mere sensations when looking at objects that are before our eyes as *'empirical intuition'*. In sense-perception - as has just been indicated - there is always an empirical part and a pure part. The pure part is always a priori. It is this pure part that contributes something to the content of a synthetic judgment a priori. Although all mathematical theorems are synthetic, as KANT believes, because they depend on constructions carried out in intuition, he is convinced that only the pure part plays any role. Thus, according to KANT, although all mathematical theorems are synthetic, they are nevertheless synthetic a priori.

These considerations lead to the following criterion for the a priori character of a synthetic judgment (cf. KrV: A713-714, B741-742): *A synthetic judgment that, in its formulation and in the proof of its truth, uses only terms and concepts that can be constructed in pure intuition is a judgment a priori.*

These considerations indicate quite clearly that KANT's theory of 'synthetic judgments a

[5] This innate disposition of our mind to 'pure sense-perception' is related to the 'direct mental perception of the ideas innate to us', which DESCARTES calls 'intuition'. Similarly, KANT*'s* 'a priori' is related to the Cartesian 'innate', but is also different from it.

priori' depends on work done by TSCHIRNHAUS. In fact, a concept (according to KANT, KrV: B741-742) is *constructible in pure intuition* if it has a *genetic definition* in the sense of TSCHIRNHAUS, cf. Chap. 10 (cf. also KANT/JÄSCHE *'Logik'* (1800), §106).

The concepts of a *straight line* and of a *circle*, for example, can be constructed in pure intuition (as KANT claims, KrV: B287), since the usual procedures for the construction of straight lines and circles (cf. Chapter 10), already indicated by HERON, are constructions that can be carried out in our intuition,

> *"aber völlig a priori, ohne das Muster dazu aus irgendeiner Erfahrung geborgt zu haben"*
> [(...) but completely a priori, without having borrowed the pattern from any experience.]
> KANT: KrV: A713, B741.

In particular, the concepts of a triangle, an equilateral triangle, a rectangle, a square, a parallelogram, etc., can be constructed in pure intuition, since they all have genetic definitions and the constructions of all these objects, which are to be found in EUCLID's *'Elements'*, can be carried out *'without having borrowed the pattern for it from any experience'*, i.e., without having to rely on physical properties (such as gravity, friction, adhesion, uniform velocity, etc. - cf. Chapter 8).

12.8 The a priori character of geometrical judgments

KANT claims that all geometrical postulates and all geometrical theorems from the first four books of EUCLID's *'Elements'* are *synthetic judgments a priori.* First of all, they are all synthetic judgments, as already shown above in the example of the *Triangle-Sum-Theorem.* They all are also a priori, since points, circles, straight lines, etc., and right angles (according to KANT) are *"ursprüngliche Vorstellungen"* (i.e., *'original representations'*), and the constructions of the required figures can be carried out in intuition, *"but completely a priori, without having borrowed the pattern for this from any experience"* (see above, cf. also KrV: B180). All postulates and all theorems from the first four books of EUCLID's *'Elements'* are therefore (according to KANT, KrV*: B16*) *synthetic judgments a priori*.

But the geometrical propositions that can be proved only through the method of directed insertion (νεῦσις, inclinatio) are not a priori. Nor are they found in the *'Elements'* of EUCLID. - Of course, all theorems whose proofs use mechanical curves (in the sense of DESCARTES, cf. Chapter 8) are a posteriori. Unfortunately, KANT did not mention these theorems. - However, let us briefly mention a few examples.

(1) The theorem that every angle can be divided into three equal parts (theorem of angle trisection) is a posteriori. It cannot be proved with the elementary means of EUCLIDean geometry, since GAUSS showed, in his *'Disquisitiones Arithmeticae', VII, §§ 365-366*, that the angle of 3° cannot be divided into three equal parts with ruler and compass (otherwise, a regular polygon of 360 sides could be constructed with compass and ruler, but $360 = 5 \cdot 2^3 \cdot 3^2$, and hence $\varphi(360) = 2^5 \cdot 3$ is not a power of 2). But, according to ARCHIMEDES (as shown in his *'Liber Assumptorum'*, VIII), any angle can be divided into three angles of equal size using the method of *'directed insertion'* (νεῦσις, inclinatio). Here, a line of given length must be inserted into a certain configuration

of circles and straight lines in such a way that its end points lie on certain lines and, at the same time, its direction passes through a certain point. Such an 'insertion' cannot be executed completely exactly! It takes place in the area of the empirically perceptible, and the final position of the inserted line, if its final position is found at all, cannot be described in elementary geometric terms.

(2) Many regular *n*-gons (i.e., regular polygons with *n* sides) can be constructed with compass and ruler, e.g., regular *3*-gons, *4*-gons, *5*-gons, *6*-gons, *8*-gons, *10*-gons, ... But, according to ARCHIMEDES, regular *7*-gons and regular *9*-gons can only be constructed with the method of *'directed insertion'*. The propositions about the existence of regular *7*-gons and the existence of regular *9*-gons are, hence - as explained above in (1) - synthetically a posteriori.

(3) FRANK MORLEY's (1860-1937) trisector-theorem, which states that, in any triangle, the three points of intersection of the adjacent angle trisectors form an equilateral triangle, is also not a synthetic judgment a priori, but *a synthetic* judgment *a posteriori.*

12.9 The a priori character of arithmetical judgments

We have explained above why KANT considered a statement like *7 + 5 = 12* to be a *synthetic judgment.* His proof was based on the concept of space and time. If, by 'intuition', one were to understand it as 'empirical intuition', then the proof would only show that *7 + 5 = 12* is a *synthetic judgment a posteriori.* But KANT claims that only that part of the intuition that is free of empirical sensations needs to be referred to here, and can therefore be called *'pure intuition'.* He argues as follows.

7 + 5 = 12 is indeed a judgment that refers to numbers of empirically perceptible objects, but to know that *7 + 5 = 12* does not require an empirically perceptible illustration. The material properties of *7* things and another *5* things added together do not matter. The fact that *7 + 5 = 12* can be seen from intuition, but only *"the pure form of sensual intuition"* (KANT: KrV: B34) plays a role. In KANT's terminology, this means that the sum of *7* and *5* can be found in a purely *schematic* way and does not require any empirical confirmation. Since only the underlying pure form of intuition is required, the judgment is *a priori.*

12.10 Discussion

KANT has largely achieved his goal of redefining the scope of our knowledge a priori - at least, in the field of mathematics, as we have shown. But we must make the restriction that this applies only to the conception of the nature of mathematics as it was understood in KANT*'s* time. KANT has not fully achieved the objective, because the 'pure intuition' he described is difficult to grasp, and it is unclear how far it reaches and whether it exists at all.

In KANT*'s* view, intuition is *"a source of knowledge, equal to thinking"*, as HERMANN COHEN aptly put it in his treatise on *'Das Prinzip der Infinitesimalmethode und seine Geschichte'* (*The Principle of the Infinitesimal Method and its History*) (1883, #21, op. cit.). KANT tried to determine that part of the intuition that is relevant to mathematics, namely, what he called 'pure intuition'. This was meant to ensure that, for example. the

straightness of straight lines, which had not been conceptually defined since antiquity, but was always intended in the practice of geometry, was available to mathematical reason. It should be allowed, for example, to make use of the straightness of "straight lines" in a descriptive sense, although "straightness" was undefinable. This, of course, also had the consequence that, in a geometry that allowed for not only thinking, but also (pure) intuition as one of the *"sources of knowledge"*, EUCLID's parallel postulate could be understood as true and valid (in the space of the real world).

For KANT, geometry is *a science that determines the properties of space synthetically and yet a priori* (*"eine Wissenschaft, welche die Eigenschaften des Raumes synthetisch und doch a priori bestimmt",* KrV: B40). The truth of the Euclidean postulates results (according to KANT) directly from their intuitive evidence (KrV: B204, B760-761). They are necessarily valid, since they are (according to KANT) the laws of pure, inherent intuition. It follows that the axioms of Euclidean geometry are *apodictically certain* (KrV: B64, and *'Prolegomena'* §§ 6-10). Consequently, Euclidean geometry would be the only possible geometry.

KANT also used 'pure intuition' to explain the applicability of mathematics to natural sciences.

> *"Die Geometrie legt die reine Anschauung des Raumes zugrunde,"*
> [Geometry is based on pure intuition of space,]

it says in KANT's *'Prolegomena zu einer jeden künftigen Metaphysik'* (*Prolegomena to any future metaphysics*), in §10. In the *'Critique of Pure Reason'*, KANT calls geometry a science that, synthetically, and yet a priori, determines the properties of space (KrV, B40). In KANT's opinion, pure reason is the form of every empirical intuition (KrV: B206). There is therefore a direct relationship between mathematics and the natural real world, and it is this relationship that makes the application possible. In his dissertation *'De mundi sensibilis atque intelligibilis forma et principiis'* (§ 15, E), KANT wrote,

> *"Natura itaque geometriae praeceptis ad amussim subiecta est, quoad omnes affectiones spatii ibi demonstratas, non ex hypothesi ficta, sed intuitive data, tanquam condicione subiectiva omnium phaenomenorum, quibus unquam natura sensibus patefieri potest."*
>
> [Nature is thus subject to all the laws that are proved in the geometry of space to the most exact (ad amussim), not on the basis of fictitious geometric axioms, but on the basis of a clearly given precondition as a subjective condition of all phenomena, through which nature can reveal itself to the senses at all].

Soon after 1800, mathematicians were no longer willing to agree with the diagnosis presented by KANT. They had to admit that neither geometry nor arithmetic had been developed in a satisfactory way since antiquity, but *'mathematics'* should be teachable and learnable (cf. Chapter 1), as well as rationally controllable. Mathematics should therefore not be based on concepts that, like the concept of *'intuition'*, are *subjective* and can hardly, or perhaps not at all, be analysed rationally.

This led to the insight that it would finally be time to lay the foundations of all mathematical theories deeper and, in doing so, to dispense entirely with intuition in proofs (!). Of course, sense-perception should continue to be a motivating factor in the formulation of mathematical concepts and theorems and proofs, but verification and proof should only

be based on rationally controllable processes. As a result, *'thinking in concepts'* was to take precedence over *'sense-perception'* in mathematics.

But what should be the foundation? - Mathematicians since antiquity had never seriously considered the foundations of their discipline. Only now, after KANT caused a sensation with his theses, did mathematicians realize how unclear they had left many things! The clarification of all of these problems occupied them throughout the 19th century and into the 20th century. This was a long period of *'errors and confusion'* (*"Irrungen und Wirrungen"*) - we will talk about this in the following Part III.

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Part III

Philosophy of Mathematics in the 19th and early 20th Century

The turn of the 18th to the 19th century brought great changes in the political and cultural order of Europe.

In the fields of the arts, philosophy and sciences, *rationalism* lost its general validity. More than ever before the *psychic-irrational* began to unfold its effects. In literature and music, around 1775, the movement of *'Sturm und Drang' (Storm and Stress), the* forerunner of *Romanticism*, began, which instead of pure, objective reason placed subjective psychic phenomena at the centre of interest.

Parallel to these changements, fundamental changes also began to take place in mathematics and in the philosophy of mathematics. The views valid until then were no longer tenable due to KANT'S critical analysis (1781/1787). But KANT'S own view was not tenable either. In order to arrive at a convincing standpoint, one had to work on deepening the foundations of mathematics. For this purpose, logic was formalized (GEORGE BOOLE, GOTTLOB FREGE, CHARLES SANDERS PEIRCE, GIUSEPPE PEANO, BERTRAND RUSSELL, DAVID HILBERT, ALFRED TARSKI and others) and the fundamental concepts of *'sets and classes'* were finally investigated from a philosophical and a mathematical point of view (BERNARD BOLZANO, GEORG CANTOR, ERNST ZERMELO, FELIX HAUSDORFF and others). The efforts were crowned with great success. They led not only to an enormous expansion of mathematics as a whole, but also to new views on the ontological status of mathematical objects and the epistemological status of mathematical theories.

These new concepts are linked to the keywords *Constructivism, Formalism, Logicism, Nominalism, Platonism, Psychologism, Structuralism,* and *Kantian philosophy of Criticism,* etc. We will report on the most important views from this list in the following Chapters 13 - 20.

Chapter 13
Psychologism in Mathematics

> *"Die Seele ist ein verdächtiges Mondscheingespinst und -gespenst"*
> [The soul is a suspicious moonlight spun and ghost.]
> THOMAS MANN*: 'Zauberberg', Werke II, Berlin 1955, p. 355.*

From the early 18th century onwards, it was often held that *psychology* is a *'general science of the mind'* ("*eine allgemeine Wissenschaft des Geistes*", as it was called by WILHELM WUNDT, *'Logik',* I, p. 1, op. cit.), and therefore the basis of all philosophy, logic and mathematics. In the 19th century, this view was even adopted by many mathematicians. It became customary in the introductory chapters of textbooks to introduce the basic mathematical terms in the terminology of psychology. It was believed that the *soul* (or *mind*) was capable of *producing* mathematical objects and that the laws of logic, arithmetic, geometry, etc., could be traced back to *psychic laws.*

But, in the course of the 19th century, the first voices to resist these tendencies began to appear. Some mathematicians and logicians appeared who did not like the interference of psychology in logic and mathematics at all and who tried to push back against the *pernicious intrusion of psychology* ("*den verderblichen Einbruch der Psychologie*"), as it was called by GOTTLOB FREGE in the first volume of his *'Grundgesetze der Arithmetik'* (1893, p. XIV, op. cit.). JOHANN FRIEDRICH HERBART (1776-1841) wrote in his Introduction to Philosophy:

> "*In der Logik ist es nothwendig, alles Psychologische zu ignorieren.*"
> [In logic it is necessary to ignore everything psychological.]
> J. F. HERBART*: 'Einleitung in die Philosophie'*, 1813, p. 23.

FREGE even wrote, in his *'Grundlagen der Arithmetik'* (1884, p. XVIII, op. cit.):

> *"... die Psychologie bilde sich nicht ein, zur Begründung der Arithmetik irgend etwas beitragen zu können."*
> [...psychology should not pretend to have anything to contribute to the foundation of arithmetic.]

This *'pernicious intrusion'* of psychology into mathematics was referred to, from 1870

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U. Felgner, *Philosophy of Mathematics in Antiquity and in Modern Times*,
Science Networks. Historical Studies 62, https://doi.org/10.1007/978-3-031-27304-9_13

onwards, with the polemically-intended expression *'psychologism'* (cf. M. KUSCH, op. cit., and V. PECKHAUS, op. cit.). One of the most astute and influential critics of psychologism was EDMUND HUSSERL, in his *'Logische Untersuchungen, I'*, op. cit.

The label *'psychologism in mathematics'* refers to the view that mathematical objects are created by the human soul and that mathematics and logic are based on laws that psychology is tasked to investigate.

When the *soul, mind* or *thinking* is spoken of in a mathematical discourse, it is not always considered to be psychologism. It is only so when, *'with the methods and results of psychology* [...] *a claim for justification is raised'*, as it was formulated by HENNING PEUCKER (*"wenn mit den Methoden und Ergebnissen der Psychologie ein Begründungsanspruch ... erhoben wird"*, H. PEUCKER op. cit., p. 98), and, in particular, *when the soul (the mind) is considered as the creator of the mathematical objects.* The following statement by FREGE may illustrate this:

> "*Man nennt den Äquator eine gedachte Linie, aber es wäre falsch, ihn eine erdachte Linie zu nennen; er ist nicht durch Denken entstanden, das Ergebnis eines seelischen Vorgangs, sondern durch Denken erkannt, ergriffen.*"
>
> [The equator is called a line we can think of, but it would be wrong to call it a line created by thinking; the line is not the result of a mental or a psychical process, but recognized, grasped by thinking.]
>
> GOTTLOB FREGE: *'Die Grundlagen der Arithmetik'*, op. cit., p. 35.

In this chapter, we want to show how the *psychologism* mentioned above could come about in mathematics and logic. In particular, we want to examine whether it was justified to speak of a *'pernicious intrusion of psychology'*.

13.1 Psyche, anima, mind and soul

The century of the Enlightenment (roughly from 1680 to 1780), in which great hopes were placed in the power of reason and rationality, hopes that were not always fulfilled, was followed by a period in which the human *soul* (the *psyche*) became the focus of attention. During this period, in literature, music, painting, philosophy and even in the sciences, directions emerged in which the psyche was seen as a richer and more fruitful treasure than reason.

The word *'psyche'* (ψυχή) is usually translated into English with the word *'soul'*. However, this translation is not quite accurate.

In Greek, ψυχή is 'the breath', 'the exhalation', and therefore, in general, 'the principle of life'. From this, ψυχή was seen as the faculty of *perception and thinking*. PLATO, for example, states, in his *'Timaios'* (30b2-6), that the Creator of the world endowed the ψυχή with spirit and reason (νους). For ARISTOTLE, ψυχή is that thing upon which we live, perceive and think (*'De anima'*, I 1, 402b13 & II 2, 414a13), i.e., the cause (αἰτία) and the principle (ἀρχή) of the living body. ARISTOTLE speaks of the 'thinking soul', whose 'divine substrate', the νους, is added to the body at the birth of a human being *('De generatione animalium'*, II 3, 736b28). According to ARISTOTLE, ψυχή has a 'nourishing soul' ('*anima vegetativa'*), a 'sensual soul' ('*anima sensitiva'*) and a 'thinking soul' ('*anima rationalis'*), whose substrate is the mind, the spirit.

In Latin, ψυχή is translated as *'animus'*. The word is related to the Greek *'anemos'* (ἄνεμος, 'wind'). *'Animare'* means 'to blow' and, in the figurative sense, also 'to animate', 'to inspire'. From this, the word *'anima'* (the 'breath') and the verbal noun 'animus' (the 'wind') are derived. But the word *'anima'* is also used in the figurative sense of 'vital strength', 'vital principle', 'vitality', 'soul'.

In German, ψυχή is translated as *'Seele'*, from which the Anglo-Saxon word *'soul'* is derived. Etymological dictionaries give the origin of these words as *'the native (or descent) of the sea'*. ('Sea' is the Anglo-Saxon translation of the German word 'See'.) This means that certain seas were considered by the Germanic tribes in ancient times as the residence of souls before birth and after death. The word *'soul'* is no longer used today in this context, but religious ideas are still contained in the word.

It follows from these considerations that, although the words *ψυχή, anima, Seele* and *soul* mean largely the same thing, they do not mean exactly the same thing. Therefore, in the following, we will prefer the Greek word ψυχή, and we will always understand the word 'soul' in the sense of 'thinking soul' (as *'anima rationalis'*), that is, as a 'soul' endowed with spirit and reason.

13.2 The role of the psyche in ancient mathematics

The abilities of the *soul* (as a 'thinking soul') had already been referred to in the field of mathematics and the philosophy of mathematics by the philosophers of antiquity. Thus, PLATO assigned the human *soul* the role of remembering everything it saw before entering worldly life, and therefore gave it an important place in the knowledge of mathematical facts (cf. Chapter 2). PROCLUS (ca. 411-485) even considered the human soul as the creator of mathematical objects. He wrote:

> Ψυχὴν ἄρα τὴν γεννητικὴν ὑποθετέον τῶν μαθηματικῶν εἰδῶν τε λόγων. …
>
> [The soul is thus ... the creator of mathematical forms and concepts. But if the soul grasps the archetypes in its essence and gives them existence and the creations are emanations of the forms already present in it, then we are in PLATO's company with such an attitude and would like to have found the real kind of Being of mathematical science.]
>
> (Translated from *Procli Diadochi in Primum Euclidis Elementorum Librum Commentarii ex recognitione* GODOFREDO FRIEDLEIN, Leipzig 1873), p. 13.)

According to PROCLUS, the exactness of mathematics cannot be justified by sensual perception, but only by the achievements of the soul.

For AURELIUS AUGUSTINUS (354-430), it is the human, rational soul that asks the wisdom of God for the knowledge of mathematical truths (cf. Chapter 8).

At the end of the Middle Ages, NIKOLAUS OF KUES (NICOLAUS CUSANUS, 1401-1464) agreed with PROCLUS' view. In his writings *'De beryllo'* and *'De ludo globi'*, it is stated that the 'intelligent soul' (*'anima rationalis'*) is the creator of arithmetic and geometry and that the soul would unfold (*explicare*), produce (*fabricare*) and bring forth (*procedere*) out of itself the numbers and the geometrical points, lines and surfaces (CUSANUS, *'De ludo globi'*, II, 93, and *'De beryllo'*, XXXIII, 55-56).

In order to avoid misunderstandings, it should be emphasized that the human mind is

obviously able to create mathematical *theories* by writing up axiom systems and Definitions, but it is more than doubtful as to whether the human mind is also able to create mathematical *objects*. And if the human mind, or the human soul, were to be able to create such objects, a great problem would arise as to what the ontological status of these objects would be. It was JOHN LOCKE who was perhaps the first to express himself in detail on this problem.

13.3 The emergence of psychologism in the modern age

Through his influential work *'An Essay Concerning Human Understanding'* (1690), JOHN LOCKE (1632-1704) was considered by many as the founder of psychologism. EDMUND HUSSERL (cf. *Husserliana XXIV, p. 206*-207) even called him the *'archfather of modern psychologism'* (*"Erzvater des modernen Psychologismus"*).

LOCKE*'s* psychologism results from his basic epistemological position that everything that the senses perceive is also perceived by the soul. It is the *soul* (or the *mind*) that forms *'ideas'* (*'Vorstellungen'* in German) from all its perceptions (LOCKE, *'Essay'*, book II, Chapter I, §3) and is able to go on creating *'mental objects' ('Gedankendinge'*) through reflection. In his *'Essay'* (op. cit., book II, chapter XI, §9), this is explained, with a few more details, as follows.

LOCKE claims that *"the mind makes (...) particular ideas received from particular objects"* (LOCKE, II, XI, §9). Out of all these *particular ideas*, the mind creates *"general ideas"* through the process of *'abstraction'*. In this process, all of those properties that are not typical for all objects under consideration (and hence are only *"concomitant",* as LOCKE says[1]) are ignored, i.e., are taken away, are 'abstracted'. What remains in the mind are the *"general ideas"*, i.e., ideas of objects with only 'typical' properties. These are objects that are obtained through reflection and are treated as mental objects, i.e., objects created by the soul. They are treated as *'representatives'* of the corresponding class of similar objects under consideration. LOCKE calls them *"naked appearances in the mind"*.

It was a mistake by LOCKE to believe that the process of creating 'general ideas' could always lead to 'general objects' of the same kind. The 'general idea' of a triangle, e.g., cannot represent triangles (i.e., cannot be a special object in the class of all triangles), because it cannot simultaneously be neither acute-angled, nor rectangled, nor obtuse-angled (cf. our Chapter 9, section 9.3). LOCKE's mistake does not follow from his doctrine of 'abstraction', although it differs considerably from the Aristotelian concept of 'abstraction' (cf. our Chapter 3). The mistake is contained in LOCKE's doctrine of 'general ideas' as representatives for classes of objects of the same kind. To assume that the soul (or the mind) is able to create mental objects might be rather dangerous. This, however, was not observed any earlier than at the end of the 19th century.

According to LOCKE, the natural numbers (*'Essay'*, II, XVI, §§ 1-8), the geometrical objects (*'Essay'*, IV, VIII, § 8) and all other mathematical objects are creations of the soul (or the mind), and all of these mental objects are created as *'general ideas'* in the way indicated above. We reported on this in Chapter 9.

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[1] The word *'concomitant'* is LOCKE's translation of the corresponding word συμβεβηκός that ARISTOTLE used. Its Latin translation is *'accidens'*. (cf. Chapter 7, Footnote 2).

LEONHARD EULER (1707-1783) contributed to LOCKE's psychologism becoming acceptable in mathematics. In the 100th letter of his *'Letters to a German Princess'* (St. Petersburg 1768-1772), written in Berlin in 1761, he followed LOCKE by calling mathematical objects *'creations of the soul'*. He begins his letter as follows (influenced by reading CICERO and also using LOCKE's terminology, such as *"Empfindungsidee"/'idea of sensation', "allgemeine Idee"/'general idea',* etc.):

> *"Die Sinne stellen uns nur Gegenstände vor, die in der Tat außer uns existieren, und die Empfindungsideen beziehen sich alle auf diese Gegenstände; aber aus diesen Empfindungsideen bildet sich die Seele eine Menge anderer, die zwar von jenen ihren Ursprung haben, aber doch keine wirklich existierenden Dinge mehr vorstellen. Wenn ich zum Beispiel den vollen Mond sehe und meine Aufmerksamkeit einzig und allein auf seine Figur hefte, so bilde ich mir die Idee der Ründe, allein ich kann nicht sagen, daß die Ründe vor sich selbst existiere. Der Mond ist zwar rund, aber die runde Figur existiert nicht besonders oder außer dem Monde."*
>
> [The senses only imagine objects that do indeed exist apart from us, and the ideas of sensation all relate to these objects; but from these ideas of sensation the soul forms a multitude of others, which, though originating from those, no longer imagine things that really exist. For example, when I see the full moon and focus my attention solely on its figure, I form [in my soul] the idea of roundness, but I cannot say that roundness exists by itself. The moon is round, but the round figure does not exist in a special way or outside the moon.]

At the sight of the full moon, the soul forms, through *'abstraction or separation'* (*"Abstraktion oder Absonderung"*), the *'idea of roundness'*. EULER writes that, in the natural real world, there is no object that is only round and has no other properties. Such an object *'exists only in my soul'* (*"es existiert nur in meiner Seele"*), as it is stated in the letter somewhat later. More generally, mathematical objects are creations of the soul. The place of their existence is the human soul.

Finally, in the 19th century, it became common practice to consider the human soul as the creator of mathematical objects. The 'creation' ('Schöpfung') of real numbers by RICHARD DEDEKIND (1831-1916) is a prominent example.

13.4 DEDEKIND's creation of irrational numbers

The first attempt to give *'a really scientific foundation'* (*"eine wirklich wissenschaftliche Begründung"*) for the concept of a real number was undertaken by RICHARD DEDEKIND in 1872, with the publication of his booklet *'Stetigkeit und irrationale Zahlen'* ('*Continuity and irrational numbers'*), cf. DEDEKIND's *'Werke'*, Vol. 3, pp. 315-334, op. cit. Everything in that booklet is well-done and mathematically fine, up to a single interference by psychologism. We will explain this below.

DEDEKIND presupposes a well-founded arithmetic of rational numbers. Let $\mathbb{Q}$ be the set of all rational numbers. An ordered pair of disjoint subsets A and B of $\mathbb{Q}$ is a *'cut'* (*"Schnitt"*) in $\mathbb{Q}$, on condition that A is an initial segment, B is an end-segment of $\mathbb{Q}$ and $A \cup B = \mathbb{Q}$. Such a cut is a *'gap'* (*"Lücke"*), if there is no rational number that separates A and B, i.e., if A has no last and B no least element. From $\mathbb{Q}$, DEDEKIND obtains the set $\mathbb{R}$ of

all real numbers by inserting a new number into each gap. The question as to what these 'new numbers' are is answered with reference to a psychological act. In fact, DEDEKIND wrote (p. 325 in vol.3 of his "Works"):

> *"Jedesmal nun, wenn ein Schnitt (A,B) vorliegt, welcher durch keine rationale Zahl hervorgebracht wird, so erschaffen wir eine neue, eine irrationale Zahl α, welche wir als durch diesen Schnitt (A,B) vollständig definiert ansehen; wir werden sagen, daß die Zahl α diesem Schnitt entspricht, oder daß sie diesen Schnitt hervorbringt."*
>
> [Whenever, then, a cut *(A,B)* is given, produced by no rational number, we create a new number, an irrational number *a*, which we consider as completely defined by this cut *(A,B)*; we shall say that the number *a* corresponds to this cut, or that it generates this cut.]

FREGE made the following comment concerning this statement (in his *'Grundgesetze der Arithmetik'*, vol. II, 1903, p. 141):

> *"In diesem Schaffen liegt der Kern der Sache. (...) Es handelt sich um die Frage, ob ein (solches) Schaffen überhaupt möglich (ist)."*
>
> [The 'creation' here is the nucleus of the whole thing. ... The question is whether such a creation is at all possible.]

However, in a letter to HEINRICH WEBER (1842-1913) dated January 24, 1888 (cf. DEDEKIND, *'Werke'*, op. cit., vol.3, pp. 488-490), DEDEKIND explained his act of creation rather surprisingly, as follows:

> *"(...) so möchte ich doch rathen, unter der Zahl (...) etwas Neues zu verstehen, was der Geist erschafft. Wir sind göttlichen Geschlechts und besitzen ohne jeden Zweifel schöpferische Kraft nicht blos in materiellen Dingen (...). Du sagst, die Irrationalzahl sei überhaupt Nichts anderes als der Schnitt selbst, während ich es vorziehe, etwas Neues (vom Schnitte Verschiedenes) zu erschaffen, was dem Schnitte entspricht, und wovon ich sage, daß es den Schnitt hervorbringe, erzeuge. Wir haben das Recht, uns eine solche Schöpfungskraft zuzusprechen. (...)"*
>
> [(...) be advised, that by (irrational) number there should be understood (...) something new, which the mind creates. We are of divine species and without any doubt possess creative power not only in material things ... You say that the irrational number is nothing else than the cut itself, whereas I prefer to create something new (different from the cut), which corresponds to the cut, and which produces the cut, as I would say. We have the right to claim such a creative power. (...).]

It seems that DEDEKIND is following the doctrine of LOCKE here that, through reflection, the soul (or the mind) would be able to create new numbers that fill the gap. This is problematic, since the existence of the resulting objects would then depend on subjective thinking. There would be no problem, however, if the subjective process of thinking could be replaced by an objective procedure, e.g., by a reconstruction of the desired objects on the basis of an axiom system of sets, such as ZSF-Set Theory, or NBG-Class Theory (cf. Chapter 15).

Fortunately, it is possible, and even quite easy, to remedy DEDEKIND's unfortunate appeal to the creative power of the soul by following WEBER's proposal to take the cuts themselves as the new irrational numbers (or more precisely, to let the cuts *play the role* of

irrational numbers). This is how it is currently done everywhere, and, in doing so, the whole construction of the field of real numbers *R* as carried out by DEDEKIND will turn out to be convincing and without any default.

Another work by DEDEKIND also contains an unfortunate appeal to the creative power of the soul. This is his 'creation' of the system of natural numbers, as carried out in his essay *'Was sind und was sollen die Zahlen?'* from 1888 (cf. DEDEKIND's Collected Mathematical Works, Vol. 3, pp. 335-391). However, the only critical point here is theorem 66 (op. cit., p. 357), where DEDEKIND refers to his own *"Gedankenwelt"* (meaning the realm of all his own possible thoughts). But, on the basis of an axiomatic Theory of Sets, the whole undertaking can be carried out totally convincingly without any contact with psychologism. (For more details on this subject, see Chapter 19.)

Finally, let us mention that DEDEKIND's celebrated theory of *'Ideals'* in Algebraic Number Theory, which is set up with the aim of obtaining unique factorization, is free of any psychologism. The definition of these *'Ideals'* is totally set-theoretical. DEDEKIND invented his *Theory of Ideals* to replace ERNST KUMMER's theory of *'ideal numbers'* from 1847, because KUMMER was unable to say what his 'ideal numbers' were (cf. DEDEKIND's Werke, Vol. 3, p. 268).

13.5 On the creation of natural numbers

What are natural numbers? The definition given by EUCLID in his *'Elements'*, book 7, is well-known: *'A number is a multitude made up of units'*. Unfortunately, this is not a very convincing definition, since it refers to a certain process of abstraction, which is not made explicit. It seems that *'units'* are objects that do not have any special properties besides their being. Do such objects exist, and, if so, are multitudes of such objects also objects? Many mathematicians in the 19th century had the idea to interpret the kind of abstraction that is relevant here as a process of abstraction à la LOCKE. GEORG CANTOR, for instance, in 1895, defined the *'cardinal number'* of a (finite or transfinite) set *M* as

> *"... den Allgemeinbegriff, welcher mit Hilfe unseres aktiven Denkvermögens dadurch aus der Menge M hervorgeht, daß von der Beschaffenheit ihrer verschiedenen Elemente m und von der Ordnung ihres Gegebenseins abstrahiert wird."*
>
> [... the *general idea* which, with the help of our active faculty of thought, arises from the set *M* by abstracting from the nature of its various elements *m* and from the order in which they are given.]
>
> G. CANTOR: *'Gesammelte Abhandlungen'*, p. 282.

This is clearly a process of abstraction à la LOCKE, which should lead to an object (!) that is created by the subjective *'active faculty of thought'*. The object arises from a Lockean *'general idea'* (*'Allgemeinbegriff'*). Since the creation of a single cardinal number by an *'active faculty of thought'* is already problematic, the simultaneous creation of the class of all finite and all infinite cardinal numbers (Alephs) is even more questionable (cf. MICHAEL HALLETT: *'Cantorian set theory and limitation of size'* (1984, pp. 120-142, op. cit.).

The process of Lockean abstraction is applied here by CANTOR to transform the individual elements of the given collection (or set) through an act of abstraction into objects

that don't have any special properties at all, besides their 'being', and are just *'units'*, as in the case of the Euclidean *'units'*. All of these 'units' are, hence, indistinguishable, all are the same. The axiom of extensionality does not seem to be valid among these multitudes of 'units'.

RUDOLF LIPSCHITZ (1832-1903) expressed himself, in his *'Lehrbuch der Analysis'* (Bonn, 1880), in a manner similar to CANTOR in the definition of the concept of a number:

> *"Wenn man bei der Betrachtung getrennter Dinge von den Merkmalen absieht, durch welche sich die Dinge unterscheiden, so bleibt der Begriff der Anzahl der betrachteten Dinge zurück".*
>
> [If, when considering separate things, one disregards the characteristics by which the things differ, the concept of the number of things considered is left behind.]

It remains unclear what is altogether 'left behind' and what is created in the end. In individual cases, one will perhaps be able to derive commutativity, associativity and distributivity from that which was 'left behind', but these properties have only then been observed in a finite number of individual cases. This is not sufficient when it is about drawing up a theory that deals with *all* numbers!

GOTTLOB FREGE (1848-1925) turned against the psychological view of the concept of numbers because they are subjective and not objective. He wrote, in his *'Grundgesetze der Arithmetik'* (volume I (1893), p. XVIII):

> *"Man darf nie vergessen, daß die Vorstellungen verschiedener Menschen, wie ähnlich sie auch sein mögen, was übrigens von uns nicht festzustellen ist, doch nicht in eine zusammenfallen, sondern zu unterscheiden sind."*
>
> [One must never forget that the representations[2] (*'Vorstellungen'* in German), different people may have, however similar they may be, which, by the way, we are not able to determine, do not coincide, but have to be distinguished.]

In his booklet *'Die Grundlagen der Arithmetik'* (op. cit., p. XIX), FREGE complains that such a psychologistic view draws *everything into the subjective* (*"zieht Alles ins Subjektive"*), and thus abolishes objective truths.

PAUL DU BOIS-REYMOND (1831-1889) described the introduction of natural numbers, in his book *'Allgemeine Functionentheorie'* (Tübingen, 1882, p. 16), with words very similar to those of LIPSCHITZ:

> *"Die Anzahl ist also gleichsam der Rest, der in unserer Seele zurückbleibt, wenn alles, was die Dinge unterschied, sich verflüchtigt, und nur die Vorstellung sich erhält, dass die Dinge getrennt waren."*
>
> [Number is thus, so to speak, the residue that remains in our soul when all that what made the things different evaporates, and only the idea (*'Vorstellung'* in German) remains that the things were separate.]

This would somehow be true, if one only knew what it is that was supposedly left behind. All of these flowery descriptions betray that the authors are probably not very sure

[2] The problem of translation of the German term *'Vorstellung'* into English is briefly discussed by ALBERTO COFFA in his essay *'Kant, Bolzano and the Emergence of Logicism'*, op. cit., p. 30.

of their case. The caricature that the mathematician CHARLES LUTWIDGE DODGSON (1832-1898), under the pseudonym LEWIS CARROLL, drew of such abstraction processes in his children's book *'Alice's Adventures in Wonderland'* (1865/1866) may give us food for thought. He wrote:

> *„All right, said the cat; and this time it vanished quite slowly, beginning with the end of the tail, and ending with the grin, which remained some time after the rest of it had gone."*

MARTIN GARDNER remarked on this (in *'The annotated Alice'*, 1960, pp. 90-91):

> *„The phrase »grin without a cat« is not a bad description of pure mathematics."*

But one should perhaps say, a bit more precisely, that the dictum *"Grin without a cat"* is only a description of the psychologistically conceived mathematical theories from the 17th up to the 19th century.

13.6 On definition by abstraction

At the end of the 19th century, mathematicians made various attempts and efforts to eliminate psychological interferences of the type mentioned above. One such attempt was worked out by the Italian Mathematician GIUSEPPE PEANO (1858-1932). In his paper *'Le definizioni per astrazione'* (1894, op. cit.), PEANO tried to put CANTOR's unfortunate definitions of the concepts of *'cardinal- and ordinal number'* on a reliable and safe basis. Thereby, he introduced what became known as his theory of *'definition by abstraction'* or *'definition by equivalence classes'*. This is the formalization of a (slight) modification of the Lockean method of creating *'representatives'* for *'general ideas'*.

Let C be any class of mathematical objects and assume that there is an equivalence relation $\approx$ defined between the elements of C. Then, two elements a and b of C may be called *'similar'* if they are equivalent with respect to $\approx$. Through *'abstraction'*, a class D of new objects is obtained (its objects are the representatives of the various equivalence classes). Also, a new binary relation S between the members of C and the members of D is defined such that two elements a and b of C are equivalent (or 'similar') with respect to $\approx$, if and only if there is an object z in D (actually a uniquely determined object z) such that a stands in relation S to z and b also stands in relation S to z,

$$\forall a \in C \forall b \in C\ (a \approx b \iff \exists z \in D\ (aSz\ \&\ bSz)).$$

Thus, we may say that z is the *'general object'* that represents all objects that are similar to a. The object z may therefore also be called the *'similarity-type'* of a and is, hence, introduced and *'defined' by abstraction*. It is a formal analogue of Lockean abstraction. The representing object z, however, is not among the elements of C.

It seems that PEANO is convinced that - quite generally - definitions always define something and, hence, define an existing *'logical object'*. This, however, is the weak point in PEANO's theory of *'definition by abstraction'*. This theory has no saving ontological foundation. It is a theory that is based on psychologism.

This becomes apparent as soon as we translate PEANO's theory into the language of set theory. Then, D is the collection of all equivalence-classes $\{b \in C;\ a \approx b\}$ (for $a \in C$) and S is nothing more than the familiar relation of elementhood. Thus, we may write $D = C/\approx$ as it

is usually done today. But these equivalence-classes need not be 'sets'. It is possible that they are absolute infinite totalities in the sense of CANTOR, which means that the resulting collection *D* would be a collection of proper classes and, hence, not an object available in set theory.

PEANO hoped that his theory of *'definition by abstraction'* could be a suitable remedy to cure CANTOR's unfortunate definitions of *'cardinal- and ordinal numbers'*. If *C* denotes the class of all sets and ≈ denotes the equivalence-relation of equipotency,[3] then PEANO's proposal was to let the equivalence class of a set be its cardinal number. But all of these equivalence-classes are proper classes, i.e., absolute-infinite totalities, which are not sets, with the sole exception of the empty set. The remedy seems to be worse than the disease.

However, it is clear that PEANO's theory of *'definition by abstraction'* should be incorporated into CANTOR's set theory, or into its axiomatized versions, namely, ZSF or NBG (see Chapter 15). In the presence of a theory of sets, PEANO's theory works quite well and can be applied in numerous cases, e.g., in the case of extending the set ℕ of natural numbers to the set ℤ of all integers, as well as extending ℤ to the ring of all rational numbers, ℚ, and extending the field of all real numbers, ℝ, to the field ℂ of all complex numbers.

By slightly modifying the formation of equivalence classes, PEANO's theory of *'definition by abstraction'* can also be applied in numerous further cases, e.g., in the creation of (the set of) natural numbers, ℕ, and of (the set of) real numbers ℝ. The modification consists in applying both the power-set-axiom and either the well-ordering theorem (the axiom of choice) or the axiom of foundation (cf. DANA SCOTT, 1955, op. cit.; U.FELGNER, 2002, op. cit.).

For further details on the original method of *'defining objects by abstraction'*, we refer the reader to the paper by PEANO that was cited above, and also to CESARE BURALI-FORTI's lecture at the International Congress of Philosophy in Paris 1900, op. cit. as well as to the books by LOUIS COUTURAT, 1905, pp. 42-43, op. cit., and HERMANN WEYL, 1928, pp. 8-12, op. cit. For a renewed interest in PEANO's principle of abstraction and the FREGE-inspired abstraction principles, we refer you to R. T. COOK (Editor): *'The Arché Papers on the Mathematics of Abstraction'* (2007, op. cit.).

13.7 Concluding remarks

Few people in the 19th century were contemplating the problems that show up here deeply enough. Most mathematicians were glad that a new discipline, called 'psychology', was established at the universities, dealing with the 'soul'. The problems were entrusted to it and to schools of philosophy[4] with a psychological orientation. They were believed to be in good hands. The psychologists gladly accepted the assignment, and began to devote themselves to the subject of the foundations of mathematics as well. But what came out of

[3] According to DAVID HUME (1736), two sets are *equipotent* if there is a mapping that maps all of the elements of the first set onto the elements of the second set in a one-to-one fashion - see our footnote 6 in Chapter15, section 15.3.

[4] A good overview of the various psychological directions in philosophy was given by WILLY MOOG in his books, which we have listed in the bibliography. We also refer to the works of HANS PFEIL, op. cit., and MATTHIAS RATH, op. cit.

it was insufficient and was later dismissed as *'psychologism'* by the more thoroughly thoughtful mathematicians and philosophers. They rightly rejected a foundation of logic and mathematics based on psychology. The final elimination of psychological interferences was achieved by the rise of set theory in the early 20th century.

Since psychology is a science that proceeds empirically, it cannot - at least, in the case of mathematics and logic - go beyond what empiricism can achieve in those two disciplines. As we have already seen in Chapter 11, however, empiricism cannot provide a convincing foundation here.

It is correct to say that the *theories* (!) of natural numbers, of irrational numbers, of complex numbers, of KUMMER's 'ideal numbers', etc., are creations of the human mind, but it is wrong to say that the individual natural numbers, irrational numbers, complex numbers, ideal numbers, etc., are objects (!) created by the human mind, or soul.

GEORGE BOOLE, RUDOLF CARNAP, LOUIS COUTURAT (op. cit.), GOTTLOB FREGE (op. cit.), MARTIN HEIDEGGER (op. cit.), EDMUND HUSSERL (op. cit.), CHARLES SANDERS PEIRCE, BERTRAND RUSSELL and many others were convinced that arithmetic needed only logic to justify it and that logic itself could be developed without the intervention of psychology. The psychologists felt attacked and spoke of a *'logicism'*, i.e., an arrogance of the logicians in giving foundations for logic and mathematics. Whether this accusation is justified or not, we will discuss in the following chapter.

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Chapter 14
Logicism

"Die Mathematik (ist) ein (....) fortentwickelter Zweig der allgemeinen Logik."
[Mathematics is a branch of a further developed general logic]
HERMANN LOTZE: *'System der Philosophie, Teil I: Drei Bücher der Logik',* Leipzig, (1874), p. 34.

The view that mathematics is a part of logic is called *'logicism'.* This means two things:

(1) All mathematical concepts and all mathematical entities are explicitly definable using only concepts and objects from logic.
(2) All true mathematical theorems are universally valid statements from logic.

The term *'logicism'* appeared shortly after 1900, having been coined by some adherents of psychologism. It was meant polemically. Just as the logicians had previously dismissed psychology's interference in the discussion on the foundations of mathematics as *'psychologism'*, the psychologists now wanted to dismiss the interference of formal logic in the foundation of the sciences as *'logicism'.* WILHELM WUNDT, for example, used this word as early 1910, in his essay on *'Psychologismus und Logizismus'* (op. cit.).

The name became generally accepted a little later, and ultimately lost its polemical aftertaste. It was in this sense that RUDOLF CARNAP, for example, used the word in his papers (1930, 1931, op. cit.) and in his '*Abriss der Logistik'* (Springer Verlag, Vienna 1929, pp. 2-3), a book he wrote in the years 1924-1929.

As a forerunner of the logicists, one could name LEIBNIZ, who wanted to derive all of mathematics from principles of logic alone (mainly the principle of identity $A = A$ and the principle of contradiction), in particular, without any use of postulates or axioms. (We discussed this in Chapter 10.) But, in contrast to logicism, LEIBNIZ did not understand mathematical objects as logical constructs, but rather as entities that are present in our soul from birth onward.

DEDEKIND could also be described as a logicist, because, in the preface to his famous essay *'Was sind und was sollen die Zahlen?* (1888) - following LOTZE - he spoke of *'the theory of numbers* ... being *a part of logic'*. In this essay, he based his ideas on the principles of set theory, which, at that time, still belonged to logic. However, set formation is based on ontological assumptions, since sets are supposed to have the status of 'things' and such

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assumptions exceed pure logic. DEDEKIND worked very intensively on the development of set theory, but took little notice of pure logic. In this respect, DEDEKIND cannot be counted among the logicists. We mentioned DEDEKIND as one of the adherents of psychologisms in Chapter 13 and will talk in detail about him as one of the founders of structuralism in Chapter 19. (However, compare ERICH RECK, 2013, op. cit., for a slightly different view.)

Two hundred years after LEIBNIZ, GOTTLOB FREGE (1848-1925) revived the idea of basing mathematics on pure logic. In an undated letter (probably from 1903) to the American mathematician EDWARD V. HUNTINGTON (1874-1952), FREGE wrote:

> *"Ich habe mir das Ziel gesetzt, die Arithmetik auf Logik allein zu begründen."*
> [I have set myself the goal of basing arithmetic on logic alone.]
> G. FREGE: *'Wissenschaftl. Briefwechsel'*, G. GABRIEL editor, Hamburg 1976, p. 89.

In somewhat more detail, FREGE wrote, in the introduction to his famous work on the '*Grundgesetze der Arithmetik'* (Volume 1, Jena 1893, page 1):

> *"In meinen »Grundlagen der Arithmetik« (Breslau 1884) habe ich wahrscheinlich zu machen gesucht, daß die Arithmetik ein Zweig der Logik sei und weder der Erfahrung noch der Anschauung irgendeinen Beweisgrund zu entnehmen brauche. In diesem Buche soll dies nun dadurch bewährt werden, daß allein mit logischen Mitteln die einfachsten Gesetze der Anzahlen abgeleitet werden.*
>
> [In my *"Grundlagen der Arithmetik"* (Breslau 1884) I tried to make it plausible that arithmetic is a branch of logic and that neither experience nor intuition are needed to provide any proof. In this book, this will be proved by the fact that the simplest laws of numbers can be derived by logical means alone.]

Thus, FREGE claimed that arithmetic requires only a few logical principles for its exposition and nothing else. Arithmetical theorems would be, hence, *analytic*, and not *synthetic a priori*, as KANT claimed (cf. Chapter 12). In his lecture '*Über formale Theorien der Arithmetik'* (July 1885, reprinted in G. FREGE *'Kleine Schriften'*, loc. cit., pp. 103-111), FREGE wrote:

> *"Es ist keine scharfe Grenze zwischen Logik und Arithmetik zu ziehen; vom wissenschaftlichen Gesichtspunkte aus betrachtet sind beide eine einheitliche Wissenschaft."*
>
> [There is no sharp line to be drawn between logic and arithmetic; from a scientific point of view, both are an undivided/ unified science.]

BERTRAND RUSSELL (1872-1970) also contradicted KANT's views, in his *'Principles of Mathematics'* (Cambridge, 1903), and justified this with the thesis that the whole of mathematics, after all, is only one branch of logic. He wrote:

> *„Thanks to the progress of Symbolic Logic, especially as treated by Professor Peano, this part of the Kantian philosophy (namely the claim that mathematical reasoning is not strictly formal, but always uses intuitions, i.e.* a priori *knowledge of space and time) is now capable of a final and irrevocable refutation. By the help of ten principles of deduction and ten other premisses of a general logical nature, all mathematics can be strictly and formally deduced; and all the entities that occur in mathematics can be defined in terms of those that occur in the above twenty premisses ....*

The fact that all Mathematics is Symbolic Logic is one of the greatest discoveries of our age."
B. RUSSELL: *‚Principles of Mathematics'*, 1903, Chap. I, 4, S. 4–5.

RUSSELL speaks somewhat boastfully of a *"final and irrevocable refutation"* of KANT's philosophy and the supposed insight that all mathematics is nothing more than symbolic logic. We will see if RUSSELL is right about this (cf. also HILARY PUTNAM, 1967, op. cit.).

CHARLES SANDERS PEIRCE (1839-1914) also occasionally accepted logistical positions. He wrote, in his essay '*Upon the logic of mathematics',* 3.20, that the theorems of mathematics would result, in a purely syllogistic way, from general logical propositions.

In contrast to PEIRCE, however, LOUIS COUTURAT, RUDOLF CARNAP, FRANK PLUMPTON RAMSEY, JAN LUKASIEWICZ and others have explicitly declared their support for logicism. In the following, we will discuss FREGE*'s* views in detail.

14.1 FREGE*'s* logicism

FRIEDRICH LUDWIG GOTTLOB FREGE was born in Wismar/Germany in 1848. His father, ALEXANDER FREGE, was director of the 'Höhere Töchterschule' there. After his 'Abitur' on Easter 1869, GOTTLOB FREGE studied mathematics, physics, chemistry and philosophy in Jena and Göttingen. He received his doctorate in Göttingen in 1873, with a dissertation called '*Über eine geometrische Darstellung der imaginären Gebilde in der Ebene'.* He habilitated at the University of Jena in 1874, with a thesis on *'Rechnungsmethoden, die sich auf eine Erweiterung des Größenbegriffs gründen'* (i.e., Computational methods based on an extension of the concept of size). He worked as a university lecturer in Jena from 1874 to 1918. In 1879, he became titular professor and, in 1896, full honorary professor. He did not receive any other honours. In 1908, the curator of the university wrote to the 'most serene preserver of the Grand Ducal Saxonian University of Jena',

"... dass von einer besonderen Würdigung Freges abgesehen werden könne, da dessen Lehrtätigkeit untergeordneter Bedeutung sei, und ohne Vorteil für die Universität."

[... that a special appreciation of FREGE could be dispensed with, since his teaching activity was of secondary importance and without advantage for the university.]

In the meantime, FREGE has become recognized as one of the most important logicians, whose work can only be compared with the logical work of ARISTOTLE or LEIBNIZ. FREGE died 1925 in Bad Kleinen (on Lake Schwerin).

FREGE published four books and about 20 treatises on logic and the foundations of arithmetic. In 1879, his main work appeared in Halle a/S: '*Begriffsschrift, eine der arithmetischen nachgebildete Formelsprache des reinen Denkens'*. FREGE first published his logicistic thesis in 1884, in his book '*Die Grundlagen der Arithmetik'*. There, he claims (in § 87), following LOTZE, that

"... die Arithmetik nur eine weiter ausgebildete Logik (sei, und) jeder arithmetische Satz ein logisches Gesetz sei."

[Arithmetic is only a further developed logic, and every arithmetical proposition is a logical law.]

For BERTRAND RUSSELL, *"all Mathematics is Symbolic Logic".* It is noteworthy that, in

contrast to RUSSELL, FREGE formulated the logicistic thesis only for arithmetic. FREGE considered the propositions of geometry to be synthetic judgments a priori. In §89 of his *'Grundlagen der Arithmetik'*, pp. 101-102, FREGE even emphasized KANT*'s* great merit for having thus revealed *'the true essence'* (*"das wahre Wesen"*) of geometry. In his dissertation (Göttingen, 1873), he wrote, in the introduction, that

> *"... die ganze Geometrie zuletzt auf Axiomen beruht, welche ihre Gültigkeit aus der Natur unseres Anschauungsvermögens herleiten.*"
>
> [... all of geometry is ultimately based on axioms which derive their validity from the nature of our capacity of perception.]

But, even more emphatically, he advocated the logicistic thesis for arithmetic. With unprecedented thoroughness, he worked out a system of logical reasoning in numerous books and essays, and, on the basis of this system, he developed arithmetic.

FREGE must have been very bitter that hardly anyone took notice of his writings. An example may illustrate this.

Under the direction of FELIX KLEIN (1849-1925), the individual issues of the *'Encyklopädie der Mathematischen Wissenschaften'*, which was supported by the Academies of Sciences in Göttingen, Leipzig, Munich and Vienna, were published from 1898 onwards. It was decided that the first issue should deal with the 'Fundamentals of Arithmetic'. As authors, in addition to FREGE, RICHARD DEDEKIND (1831-1916), ERNST SCHRÖDER (1841-1902), OTTO STOLZ (1842-1905) or perhaps even JOHANNES THOMAE (1840-1921) or GEORG CANTOR (1845-1918) could have been considered as well. But the editors believed the topic to be rather unimportant, and had asked the Gymnasial-Oberlehrer HERMANN SCHUBERT to work on it.[1]

In a short essay (op. cit., p. 296), the co-editor FRANZ MEYER praised the contribution submitted by Mr. SCHUBERT for *'bringing out in detail the importance that psychology had meanwhile acquired in the foundation of arithmetic'.*

But FREGE was horrified when he saw SCHUBERT's report. In a small booklet, '*Über die Zahlen des Herrn H. Schubert'* (H. Pohle Verlag, Jena, 1899), he tore Mr. SCHUBERT's work to pieces. FREGE began his small booklet as follows:

> *"Es ist doch eigentlich ein Skandal, daß die Wissenschaft noch über das Wesen der Zahl im unklaren ist. Daß man noch keine allgemein anerkannte Definition der Zahl hat, möchte noch angehen, wenn man wenigstens in der Sache übereinstimmte. Aber selbst darüber, ob die Zahl eine Gruppe von Dingen oder eine mit Kreide auf einer schwarzen Tafel von Menschenhand verzeichnete Figur sei, ob sie etwas Seelisches, über dessen Entstehung die Psychologie Auskunft geben müsse, oder ob sie ein logisches Gebilde sei, ob sie geschaffen sei und vergehen könne, oder ob sie ewig sei, selbst darüber hat die Wissenschaft noch nichts entschieden. Ist das nicht ein Skandal? Ob ihre Lehrsätze von*

[1] HERMANN CAESAR HANNIBAL SCHUBERT was born in Potsdam in 1848, and died in Hamburg in 1911. He received his doctorate in 1870, in Halle, and, in 1875, he was awarded the Great Golden Medal of the Royal Danish Academy for his prize-writing *'Characteristicum der Raumkurven dritter Ordnung'.* In the highly estimated 'Sammlung Göschen', he published a book on *'Arithmetik und Algebra'.* W. BURAU expressed his appreciation for SCHUBERT's work in an essay *'Der Hamburger Mathematiker Hermann Schubert'* (Mitteilungen der Hamburgischen Math. Gesellschaft, Band 9, 1966).

jenen aus kohlensaurem Kalke bestehenden Gebilde oder von unsinnlichen Gegenständen handeln, weiß die Arithmetik nicht. (...) Die Wissenschaft weiß also nicht, welchen Gedankeninhalt sie mit ihren Lehrsätzen verbindet; sie weiß nicht, womit sie sich beschäftigt. (...) Ist das nicht ein Skandal? Und ist es nicht ein Skandal, daß eine Kette von Gedankenlosigkeiten mit Erfolg den Anspruch erheben kann, dem neuesten Stande der Wissenschaft zu entsprechen?"

[It is actually a scandal that science is still unclear about the nature of numbers. The fact that there is still no generally accepted definition of 'number' would be acceptable, if one at least agreed on the matter. But even whether the number is a group of things or a figure written on a blackboard by a human hand in chalk, whether it is something spiritual, the origin of which psychology must tell us, or whether it is a logical construct, whether it is created and can pass away, or whether it is eternal, science has not yet decided anything about that. Is that not a scandal? Arithmetic does not know whether its theorems are about those constructs consisting of carbonated lime or about nonsensical objects .... So science does not know what content of thoughts it associates with its theorems; it does not know what it is dealing with .... Isn't that a scandal? And is it not a scandal that a chain of thoughtlessness can successfully claim to correspond to the latest state of science?]

- and what follows is biting derision, corroding irony. FREGE must have been appalled that an essay

"... von solcher Flachheit es wagen könne, sich in einer Enzyklopädie der Mathematik als Blüte der Wissenschaft hinzustellen."

[... of such flatness could dare to present itself in an encyclopedia of mathematics as the flower of science.]

For HERMANN SCHUBERT (and, by the way, also for some other well-known philosophers, for example, the Hegelian KUNO FISCHER), *the number is the result of counting* (*"Die Zahl ist das Ergebnis des Zählens*").[2] FREGE derides: *'Indeed, is not also the weight of a body the result of weighing?'* - It is worth reading FREGE*'s* essay in its entirety, for FREGE's own efforts to introduce numbers as logical constructs and to construct arithmetic correctly and clearly on the basis of a formal logic are all the more clearly set off against it.

To be able to execute his program, FREGE first worked out a System of Logic. He

(1.) designed a novel symbolism for Logic,

(2.) replaced the classical classification of statements according to the scheme "subject-predicate" with the scheme "argument-function", and thereby (1879) introduced the concept of the quantifier,

(3.) and recognized (1893) that, instead of *'equality'*, *'identity'* is a logical constant that can be placed alongside the basic logical constants of *'not', 'and', 'or', 'for all'* (cf. U. FELGNER, 2020, op. cit.) - and, finally -

[2] KUNO FISCHER (1824-1907) wrote, in his work *'System der Logik und Metaphysik oder Wissenschaftslehre',* I, 2, § 93, (1852): *"Durch das Zählen entsteht die Zahl. Die Zahl ist gezählte, d.h. begriffene Größe"* [„Counting gives rise to number. The number is counted, i.e., conceptual quantity."] (reprinted in FISCHER*'s 'Philosophische Schriften'*, Volume 6 (Heidelberg 1909), p. 233).

(4.) set up the first complete Logic Calculus. (The completeness was first proved, in the case of FREGE's Propositional Logic, by PAUL BERNAYS in 1917/1918, and, in the case of FREGE's Predicate Logic, restricted to the first-order, by KURT GÖDEL in 1929/1930).

14.2 FREGE's foundation of arithmetic from the point of view of logicism

If $\Phi(x)$ denotes a concept, then $\dot{\alpha}\Phi(\alpha)$ (following FREGE) denotes the *extension* of Φ (*étendue de l'idée*, in the sense of ANTOINE ARNAULD's *'La Logique ou l'Art de penser',* I,6, Paris 1662). Today, we write $\{x; \Phi(x)\}$ instead, and read: class (or set) of all objects x that fall under the concept Φ. [3] The following axioms are postulated as valid for such concepts (we do not use FREGE*'s* notation, but rather the usual conventions of today):

(F1) *Comprehension axiom*: *For every concept* Φ, *its extension is an existing thing,* i.e.,

$$\exists y : y = \{x; \Phi(x)\}.$$

(F2) *Axiom of extensionality*: *Two extensions of concepts are identical if their underlying concepts are logically equivalent*. Hence*, for concepts* Φ *and* Ψ, *the following is true:*

$$\{x; \Phi(x)\} = \{x; \Psi(x)\} \leftrightarrow \forall x(\Phi(x) \leftrightarrow \Psi(x)).$$

The axiom of comprehension[4] states that, for each property $\Phi(x)$, its extension is an object that belongs to the domain of things that may occur as values of variables.

The two axioms (F1) and (F2) were first formulated by BERNARD BOLZANO in 1837. FREGE mentions (F1) only in the epilogue of the *'Grundgesetze'*, volume 2, p. 253. But the axiom (F2) is explicitly formulated by FREGE: it is his axiom five, V, in §20 of the *'Grundgesetze'*, volume 1, page 36. In his formulation, it looks as follows:

$$\vdash (\{x; \Phi(x)\} = \{x; \Psi(x)\}) = (\forall x (\Phi(x) = \Psi(x))),$$

because $\Phi(x)$ and $\Psi(x)$ are understood as functions, and the axiom says that the truth-values (Wahrheitswerte) of $\{x; \Phi(x)\} = \{x; \Psi(x)\}$ and $\forall x(\Phi(x) = \Psi(x))$ are the same.[5]

FREGE considered the transition from a concept to its extension to be a purely logical process. He considered concepts and their extensions to be logical constructs! Whence the

---

[3] We are obviously deviating from FREGE's notation here. In his book *'Grundgesetze der Arithmetik'*, Volume 1, Jena 1893, p. 7 & p. 15, he designated, by $\dot{\alpha}\Phi(\alpha)$, the course of value of the function $\Phi(x)$. Thus, it is possible to speak in a simple way about objects that fall under a concept, as well as about objects that do not fall under this concept. In order to make it easier for today's readers, we have used $\dot{\alpha}\Phi(\alpha)$ instead to denote the extension of the concept $\Phi(x)$, i.e., the totality of all objects that fall under the concept Φ. The subsequent definitions of the arithmetic terms have been adapted accordingly.

[4] *Comprehendere* (lat.) means: to take together, to summarize, to assemble, to embrace.

[5] See also the Appendix by ROY T. COOK in the new English, complete translation of FREGE's *'Basic Laws'*, edited and translated by PH. EBERT & M. ROSSBERG*, Oxford* 2013, op. cit.

use of extensions of concepts remains entirely within the bounds of logic, because the intersection of two extensions of concepts is nothing more than the extension of the logical conjunction of the two concepts, etc.

FREGE's program consists in developing the entire arithmetic on the basis of the two axioms (F1) and (F2) and the logic calculus. He introduces the concept of number as follows:

- Two concepts $\Phi(x)$ and $\Psi(x)$ are called *numerically equivalent* (*"gleichzahlig"*) if there is a one-to-one correspondence of the objects satisfying $\Phi(x)$ and the objects satisfying $\Psi(x)$ (*'Grundgesetze'*, § 38, page 56).
- The *number of the concept* $\Phi(x)$ [in symbols: $\#\Phi(x)$] is the extension of the concept "to be numerically equivalent to $\Phi(x)$" (*'Grundgesetze'*, § 40, page 57). In other words (if one identifies concepts by their extensions): the cardinal number of a set *M* is the class of all sets equipotent with *M.*
- *n* is a *number,* if there is a concept $\Phi(x)$ such that *n* is its number.
- *0* = the number of the concept 'not identical to itself'.

  $0 = \#(x \neq x)$ = Class of all unfulfillable concepts.
- *1* = the number of the concept 'identical to *0*' (*'Grundgesetze'*, § 41-42, pages 57-58). Thus, *1* is the class of all concepts whose number is "one":

  $1 = \#(x = 0)$ = class of all concepts that are fulfilled by exactly one object.
- The number *m* is the *direct successor* of the number *n*, if there is a concept $\Phi(x)$ and an object *a*, that falls under the concept $\Phi(x)$, so that $n = \#(\Phi(x) \,\&\, x \neq a)$ and $m = \#\Phi(x)$ ('*Grundgesetze', §* 43, page 58).

Therefore, for example, *2* is the class of all concepts that are satisfied by one object and also by another object, but by no further object. (We try to avoid speaking of *'two'* objects in the definition of the number *2*.)

The *number of the concept* $\Phi(x)$ is denoted by FREGE with an upside down handwritten character *lb* [for *libra* (lat.) = weight, pound]. We have replaced it with the typographically similar character #.

In FREGE's work, numbers are objects that carry their own meaning, such as a gold coin that reads "100 DM" is also, *per se*, worth 100 DM in terms of metal and weight, in contrast to paper money, upon which "100 DM" is only printed and that has this value only by convention (and can therefore be devalued by governments as often as they like). In the terminology of ARISTOTLE, we could say that the objects defined by FREGE are numbers *per se* (καθ' αὑτό) and not *per accidens* (κατὰ συμβεβηκός).

With a thoroughness never seen before, FREGE proves the fundamental theorems of arithmetic (for a proof sketch, see FREGE's *'Grundlagen der Arithmetik',* 1884, §§ 80-82). In particular, he proves that every number *n* has only one direct successor, which may therefore be denoted as $n + 1$. FREGE stresses, in his *'Grundlagen der Arithmetik'* (1884, § 82), that such a proposition cannot be substantiated empirically. FREGE suggests a proof of the principle of complete induction in § 80. - The structure of the field of real numbers is the subject of FREGE's 2nd volume of his '*Grundgesetze der Arithmetik'* (Jena 1903).

Until around the turn of the century (1900), one could have had the impression that the logicistic program was feasible and that the most important parts of mathematics, the arithmetic of natural and real numbers, could be built on the indisputable foundation of pure logic. The natural numbers would be *constructs of logic* and the propositions of arithmetic would be *apodictic truths,* i.e., irrefutable, logically certain statements. - The appearance of set-theoretical antinomies, however, destroyed this impression.

14.3 The appearance of antinomies

Various antinomies appeared in various mathematical disciplines around the turn of the century (1900). They were noticed by CESARE BURALI-FORTI (1897), ERNST ZERMELO (1900) & BERTRAND RUSSELL (1901/1902), JULES RICHARD (1905), G.G. BERRY (1906), KURT GRELLING (1908) and others (see also Chapter 15). FREGE probably did not hear anything about this at first. But, in June 1902, he received a letter from BERTRAND RUSSELL in which he learned that his development of arithmetic was also threatened by the appearance of antinomies. RUSSELL wrote (in German):

> RUSSELL to FREGE (16.06.1902): *"Sehr geehrter Herr College! Seit anderthalb Jahren kenne ich Ihre »Grundgesetze der Arithmetik«, aber jetzt erst ist es mir möglich geworden die Zeit zu finden für das gründliche Studium das ich Ihren Schriften zu widmen beabsichtige. Ich finde mich in allen Hauptsachen mit Ihnen in vollem Einklang (...) Nur in einem Punkte ist mir eine Schwierigkeit begegnet. Sie behaupten (S. 17) es könne auch die Funktion das unbestimmte Element bilden. Dies habe ich früher geglaubt, jedoch jetzt scheint mir diese Ansicht zweifelhaft, wegen des folgenden Widerspruchs: sei w das Prädicat, ein Prädicat zu sein welches von sich selbst nicht prädicirt werden kann. Kann man w von sich selbst prädiciren? Aus jeder Antwort folgt das Gegenteil. Deshalb muß man schliessen, dass w kein Prädicat ist. Ebenso giebt es keine Klasse (als Ganzes) derjenigen Klassen die als Ganzes sich selber nicht angehören. Daraus schliesse ich, dass unter gewissen Umständen eine definierbare Menge kein Ganzes bildet. (...)*
>
> *Mit hochachtungsvollem Grusse, Ihr ergebener Bertrand Russell."*
>
> [Dear Mr College! I have known your 'Basic Laws of Arithmetic' for a year and a half, but only now have I been able to find the time for the thorough study I intend to dedicate to your writings. I find myself in full agreement with you in all main matters ... Only in one point I have encountered a difficulty. You claim (p. 17) that a function can also form the indeterminate element. I used to believe this, but now this view seems doubtful to me, because of the following contradiction: Let *w* be the predicate, to be a predicate that cannot be predicted of itself. Is it possible to predicate *w* of itself? From each answer its opposite follows. Therefore one must conclude that *w* is not a predicate. Likewise, there is no class (as a whole) of those classes which as a whole do not belong to themselves. From this I conclude that under certain circumstances a definable set does not form a whole.
>
> Yours sincerely, Bertrand Russell.]

FREGE replied to him on June 22, 1902, with great sadness:

> *"Ihre Entdeckung des Widerspruchs hat mich auf's Höchste überrascht und, fast möchte ich sagen bestürzt, weil dadurch der Grund, auf dem ich die Arithmetik sich aufzubauen dachte, in's Wanken gerath. Es scheint danach, (...) daß mein Gesetz (V) falsch ist. ...."*
>
> [Your discovery of the contradiction has surprised me to the utmost and, I would almost

say, dismayed me, because it has shaken the basis on which I intended to build arithmetic. It seems then, ... that my law (V) is wrong .... .]

FREGE was certainly more than dismayed, for, in his earlier publications, he had covered the way his colleagues approached the construction of arithmetic with irony and ridicule. And now this had happened to him!

The first volume of FREGE*'s 'Grundgesetze der Arithmetik'* had already been published in Jena in 1893. The printing of the second volume was almost finished when FREGE received the letter from RUSSELL. FREGE added an epilogue to the 2nd volume and had it published in Jena in 1903 (at his own expense). In the epilogue, FREGE wrote:

"Einem wissenschaftlichen Schriftsteller kann kaum etwas Unerwünschteres begegnen, als daß ihm nach Vollendung einer Arbeit eine der Grundlagen seines Baues erschüttert wird.
In diese Lage wurde ich durch einen Brief des Herrn Bertrand Russell versetzt, als der Druck dieses Buches sich seinem Ende näherte. Es handelt sich um mein Grundgesetz (V). ... Ich hätte gern auf diese Grundlage verzichtet, wenn ich irgendeinen Ersatz dafür gekannt hätte. Und noch jetzt sehe ich nicht ein, wie die Arithmetik wissenschaftlich begründet werden könne, ... wenn es nicht - bedingungsweise wenigstens - erlaubt ist, von einem Begriffe zu seinem Umfange überzugehn. Darf ich immer von dem Umfange eines Begriffes, von einer Klasse sprechen? Und wenn nicht, woran kennt man die Ausnahmefälle?
Doch zur Sache selbst! Herrr Russell hat einen Widerspruch aufgefunden, der nun dargelegt werden mag. ..."

[There is hardly anything more undesirable for a scientific writer than to have one of the foundations of his construction shaken after completing his work.

I was put in this position by a letter from Mr. Bertrand Russell when the printing of this volume was nearing its end. It concerns my Basic Law (V). (....) I would have gladly renounced this basis if I had known any substitute for it. And even now I do not see how arithmetic can be scientifically founded (...) if it is not - at least conditionally - permitted to move from one concept to its extension. May I always speak of the extension of a concept, of a class? And if not, how does one recognize the exceptional cases? (…)

But to the matter itself! Mr. Russell has found a contradiction, which may now be explained .... .]

FREGE later tried to find a way out of this disaster, but he never came up with a convincing solution.

FREGE could have known about the danger of antinomies in his system some fourteen years earlier, for he had read CANTOR's 1888 treatise '*Mitteilungen zur Lehre vom Transfiniten'*, and had even written an extensive review.[6] In it, CANTOR expressly points out that only the *finite* and *transfinite* sets, but not the *'absolute-infinite'* multitudes, are available in mathematical reasoning (more on this in Chapter 15). FREGE obviously overlooked the significance of this remark. (The 'numbers' in FREGE*'s* work are 'absolute-infinite' classes (in the sense of CANTOR), but they occur in the comprehension axiom (F1)

[6] The review appeared in the Zeitschrift für Philosophie und philosophische Kritik, Volume 100 (1892), pp. 269-272, reprinted in FREGE*'s 'Kleinen Schriften'*, op. cit., pp. 163-166. A draft of this review can be found in FREGE*'s 'Nachgelassene Schriften'*, op. cit., pp. 76-80.

as values of variables, and hence are treated like ‘things’, and that means like ‘sets’.)

The antinomy that RUSSELL pointed out to him is that which has, in the meantime, become well known as the ZERMELO-RUSSELL antinomy (cf. Chapter 15):

The class of all classes which do not belong to themselves (as a whole), belong to themselves (as a whole), exactly when they do not belong to themselves (as a whole). Thus, for $a = \{x; x \notin x\}$ it follows that: $a \in a \Leftrightarrow a \notin a$.

ZERMELO had already presented this reasoning in his Göttingen lecture on set theory in the winter semester 1900/1901 - see U. FELGNER : *'Introductory Note'* in Volume I of the *‘Gesammelte Werke Zermelos’*, op. cit., pp. 167-168. RUSSELL found this reasoning, according to his own notes, between 1901 and 1902.

It is, however, noteworthy that RUSSELL's own theory of cardinal numbers is defective for exactly the same reason! In his book *'The principles of Mathematics'* (Cambridge, 1903, #111, p. 115, op. cit.), he presented his theory of cardinal numbers, claiming that it would be *"an irreproachable definition of the number of a class in purely logical terms"*.

RUSSELL defined the cardinal number of a class *M "as the class of all classes similar to the given class"*. Thus, the number *1*, for example, would be the class of all classes that contain precisely one element. But, then, e.g. $\{\{y\}; y \subseteq 1\} \subseteq 1$, which is contradictory, as can be shown by the usual Cantorian diagonal argument.[7] It is quite surprising that RUSSELL did not realize that he was making exactly the same mistake in his Chapter XI that he had found in FREGE's work and that he presented in detail in his Chapter X, pp. 101-107. Quite surprising indeed! - It was FELIX HAUSDORFF who, in his review of RUSSELL's book, was the first to observe the mistake (cf. HAUSDORFF, 1905, op. cit., pp. 123-124; see also MICHAEL HALLETT, 1984, p. 87, op. cit., and U. FELGNER: *'Der Begriff der Kardinalzahl'*, 2002, op. cit.). We will return to this subject in Chapter 15, section 15.4.

The above-mentioned antinomy did not result from FREGE's axioms of Predicate Logic, but from the axioms (F1) and (F2). Only now did it gradually become clear that the principles (F1) and (F2) do not belong to pure logic at all, because they make *ontological assertions*, namely, the *existence* of sets or classes (i.e., according to (F1), the existence of collections as things), and that such statements are not universally valid. From (F1), for example, the existence of an infinite set also results, for instance, as the extension of the concept of a ‘natural number’.

As a result of this development, we can state that *logicism* has failed, because it has taken the concept of logic too broadly and allowed contradictory principles to arise, principles that are actually set-theoretical in nature.

BERTRAND RUSSELL even deliberately defined the term ‘logic’ very broadly. In his essay *'L'Importance Philosophique de la Logique'* (Revue de Métaphysique et de Morale, vol. 19 (1911), p. 281-291), he wrote:

> *"En parlant de la »logique mathématique« je désire employer ce mot dans un sens très large: j'y comprends les travaux de Cantor sur le nombre infini aussi bien que les travaux*

[7] In fact, consider the class $A = \{\{y\}; y \subseteq 1 \ \& \ \{y\} \notin y\}$. Then, $A \subseteq 1$ and $\{A\} \in A \Leftrightarrow \{A\} \notin A$ follows as usual - a contradiction!

de Frege et de Peano."

[Speaking of 'mathematical logic' I wish to use the word in a very broad sense: including Cantor's work on infinite numbers as well as the works of Frege and Peano.]

In the three-volume work *'Principia Mathematica'*, which RUSSELL wrote together with ALFRED N. WHITEHEAD, mathematics is developed on the foundation of 'logic', but RUSSELL's ramified type theory, the axiom of infinity, the axiom of choice and the problematic axiom of reducibility are also counted among the principles of 'logic'. Nobody today would understand the concept of 'logic' in such a broad sense.

DAVID HILBERT described the failure of FREGE's logicism in his treatise '*Neubegründung der Mathematik'* (Abhandl. Math. Seminar Hamburg, Vol. 1 (1922), pp. 157-177, there p. 162) as follows:

"Frege hat die Begründung der Zahlenlehre auf reine Logik, Dedekind auf Mengenlehre als ein Kapitel der reinen Logik versucht: beide haben ihr Ziel nicht erreicht. Frege hatte die gewohnten Begriffsbildungen der Logik in ihrer Anwendung auf Mathematik nicht vorsichtig genug gehandhabt: so hielt er den Umfang eines Begriffs für etwas ohne weiteres Gegebenes, derart, daß er dann diese Umfänge uneingeschränkt wieder als Dinge selbst nehmen zu dürfen glaubte. Er verfiel so gewissermaßen einem extremen Begriffsrealismus."

[FREGE attempted to base number theory on pure logic, DEDEKIND on set theory as a chapter of pure logic: neither of them achieved their goal. FREGE had not been careful enough with the usual concepts of logic in their application to mathematics: he considered the extension of a concept to be something given without further ado, in such a way that he then believed he could take these extensions as things himself again without restriction. He thus fell into a kind of extreme conceptual realism.]

The deeper reason for the *failure of logicism* can be found in the incompleteness theorems of KURT GÖDEL (1931). They state *that mathematics has a higher complexity than pure logic.* The set *Taut*($\mathscr{L}$) of all tautologies, i.e., of all universally valid $\mathscr{L}$-sentences, is recursively enumerable, or, in other words: *semi-decidable* (if $\mathscr{L}$ is any formal language of first-order with a recursive alphabet). On the contrary, the set of all true (i.e., valid in the Standard Model) number-theoretical statements (of the 1st order) is not recursively enumerable, as the first of GÖDEL's two incompleteness theorems states. If not all true propositions of the arithmetic of natural numbers are universally valid statements of logic, then, certainly, neither are all true propositions of mathematics. *Thus, mathematics cannot be founded on pure logic alone.*

The adherents of logicism, however, only failed to achieve their goal because, on the one hand, they used the concept of logic much too broadly, even including set theory in their logic, and, on the other hand, they were a little too careless in the formulation of the principle of comprehension (F1), which became apparent in the appearance of the antinomies. It had already become clear in the early 20th century that the goals can be 'largely' achieved within the framework of an axiomatically conceived set theory. We will report on this in the following chapter.

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Chapter 15
The Concept of a 'Set'

> *"Die Mengenlehre ... (ist) ... eine der fruchtreichsten und kräftigsten Wissenszweige der Mathematik überhaupt."*
>
> [Set theory is one of the most fruitful and most powerful branches of mathematics ever.]
>
> DAVID HILBERT*: 'Axiomatisches Denken', 1917, Werke III, p. 152.*

As mathematics developed rapidly in the modern age, it became necessary to consolidate its foundations. In particular, it became necessary to define the concepts of positive, negative, real and complex numbers, free from objections, and to consider their ontological and epistemological status. The philosophers' traditional views on the foundations of mathematics were studied diligently, but were not well received.

At the beginning of the 19th century, the great works of the mathematicians of classical antiquity were recalled, and the intention was to return to them in the foundation of mathematics. In particular, their use of the *concept of a set* in establishing the concept of the number seemed to indicate that this concept should be studied carefully in order to achieve a proper foundation, not only of arithmetic, but of mathematics as a whole.

It became apparent that the contours of the *concept of a set* had to be tightened in order to achieve an efficient and, at the same time, problem-free handling of sets. We want to report on what was philosophically important in this development.

15.1 The concept of a *'set'* in classical antiquity

To describe mathematical objects and mathematical facts, the Greeks already used a word that has almost the same meaning as the modern concept of a *'set',* which is the basic concept of Cantorian set theory. They used the word *plêthos* (πλῆθος), which means, in verbal translation, *'multitudo'* in Latin and *'multitude'* in English. THALES[1] and EUCLID had already defined the concept of the *'number'* using this word. In the *'Elements'* of EUCLID, book 7, we read:

[1] According to the testimony of IAMBLICHOS in *'In Nicomachi arithmeticam introductionem liber'*, THALES had already introduced the concept of numbers in this way around 600 B.C.E.

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Ἀριθμὶς δὲ τὸ ἐκ μονάδων συγκείμενον πλῆθος.
[*Numerus autem est multitudo ex unitatibus composita.*]
[*Number is a multitude made up of units.*]

In the 9th book, theorem 20, EUCLID proves the fact that there are infinitely many prime numbers and expresses himself as follows (cf. also chapter 5):

Οἱ πρῶτοι ἀριθμοὶ πλείους εἰσὶ παντὸς τοῦ προτεθέντος πλήθους πρώτων ἀριθμῶν.
[*Primi numeri plures sunt quauis data multitudine primorum numerorum.*]
[*There are more prime numbers than any given (finite) multitude of prime numbers.*]

The words πλῆθος and *'multitudo'* are derived from *'poly'* (πολυ), resp. *'multus'* ('many'). Notice that the use of the words πλῆθος or *'multitudo'* in ancient mathematical texts is always colloquial. These words do not appear as well-defined concepts. What the Greeks understood by the noun πλῆθος can be found in ARISTOTLE and in PLOTINUS. In PLOTINUS (205-270 o.c.), we read, in his treatise *'Of numbers'* (Περὶ ἀριθμῶν):

> 'just as »multitude« (πλῆθος) is nothing more than a large number of things, or »festive gathering« (ἑορτή) nothing more than the people who flock together to enjoy the holy celebration, ... '.

A *multitude* here is merely a heap of things, just as a crowd of people (ὄχλος) is only a larger group of individual persons. From an ontological point of view, only the individuals have a *being*; the *accumulations* (or *multitudes, crowds,...*), however, are not understood as entities, and therefore have no *existence* of their own.

ARISTOTLE expressed himself about the *concepts of quantity* and *multitude* in his *'Metaphysics'* (5th book, Chapter XIII, 1020a6-11) as follows:

> 'Quantity (ποσὸν) means that which is divisible into constituent parts, every one of which is by nature some one individual thing. Thus multitude, if it is numerically calculable, is a kind of quantity; and so is magnitude, if it is measurable. »Multitude« (πλῆθος) means that which is potentially divisible into non-continuous parts; and »magnitude« (μέγεθος) that which is potentially divisible into continuous parts. '

ARISTOTLE here distinguishes the objects of arithmetic (the numerically calculable multitudes) from the objects of geometry (the measurable magnitudes).

In Greek mathematics, the word *'multitude'* (or *'set'*) is not understood as a concept or a terminus technicus, but only as an informal term with which larger surveyable accumulations of things can be addressed. These *multitudes* are not considered as *objects* or *things* themselves.

In the 17th century, ANTOINE ARNAULD and PIERRE NICOLE, in their work *'La Logique ou L'Art de Penser'* (Paris 1662), also known as *'Logique de Port Royale'*, introduced so-called *'extensions of concepts'* in Part 1, Chapter VI. An *'extension of a concept'* (*'étendue de l'idée', 'sphaera notionis'*) is the *totality* (or *multitude* or *class*) of all objects satisfying a given concept. The *'extension of a concept'* is merely a linguistic formation, and should not be understood as an entity, i.e., as an actually existing object. Only the individual objects that fall under the respective concepts are entities, and hence have (real) existence.

In the German-speaking countries, the concept of *'Menge'* (*'set'*) appeared in the early

19th century as a new mathematical concept. It was BERNARD BOLZANO *(1781-1848)*, a Bohemian-Austrian mathematician, philosopher and theologian, who introduced it. The word *'Menge'* was familiar in the German language. BOLZANO, however, provided this word with a few new marks of distinction, thus creating a new concept. He also undertook the first steps towards a systematic investigation of this concept and published his results in four volumes, in his *'Wissenschaftslehre'* (Sulzbach/Oberpfalz, 1837). Further results were published posthumously in his booklet *'Paradoxien des Unendlichen'* (Leipzig, 1851) and in his *'Größenlehre'* (Stuttgart, 1975). Through the works of GEORG CANTOR (1845-1918), the concept of *'Menge'* became a fundamental concept in mathematics. Its translation into French is *'ensemble'*, into Italian *'insieme'* and into English *'set'*.

15.2 The BOLZANO concept of a *'set'* (*'Menge'*)

The distinction between the quantities of arithmetic (the numerically calculable multitudes) and the quantities of geometry (the measurable distances, surfaces, bodies, etc.), which ARISTOTLE had made (see above), was paraphrased in the late 18th century by the Göttingen mathematician ABRAHAM GOTTHELF KÄSTNER (1719-1800) in his textbook *'Anfangsgründe der Arithmetik, Geometrie etc.'* (4th edition), Göttingen 1786, page 3, as follows:

> *"Man kann die Grösse blos als eine Menge von Theilen, als ein Ganzes (totum) betrachten; oder man kann zugleich auf die Verbindung und Ordnung dieser Theile sehen, welche ein gewisses zusammengesetztes Ding (compositum) ausmachet. In jener Betrachtung gehöret sie für die Arithmetik, in dieser für die Geometrie."*
>
> [One may consider a quantity merely as a set of parts, as a whole (*totum*); or one may consider also the connection and order of these parts, which makes up a certain composite thing (*compositum*). In that view it belongs to arithmetic, in the other view it belongs to geometry.]

Thus, a quantity consisting of parts (i.e., elements) can be considered either as a *totum* (a whole) or a *compositum* (a composite), depending on whether it belongs to arithmetic or to geometry. If it is regarded as a *totum*, then any arrangement of its parts is irrelevant; if it is regarded as a *compositum*, then the arrangement or connection of its parts is relevant (cf., e.g., RÖSLING, op. cit.).

BOLZANO took up KÄSTNER*'s* description and raised it to a definition of the terms *'Menge'* ('set') and *'Inbegriff'* (i.e. 'Verkörperung', 'reification', embodiment', an object that possesses certain qualifications, a structured set).[2] He called quantities *'Mengen'* ('sets') if the way the individual parts are connected is to be disregarded. If the kind of connection is to be considered, then they are called *'Inbegriffe'* (see BOLZANO, §§ 82-85 in the 1st volume of his *'Wissenschaftslehre'* and §4 of his booklet on the *'Paradoxien des Unendlichen'*). From an *'Inbegriff'*, a *'Menge'* arises if one abstracts from the kind of connection of the individual parts. In his *'Einleitung zur Größenlehre'*, he wrote:

> *"Inbegriffe nun, bey welchen auf die Art, wie ihre Theile miteinander verbunden sind, gar*

[2] BOLZANO quotes this passage from KÄSTNER*'s 'Anfangsgründe'* in his *'Größenlehre'*, op. cit., p. 41. That BOLZANO was well acquainted with KÄSTNER's *'Anfangsgründe'* can also be seen from the preface to his *'Beiträge zu einer begründeteren Darstellung der Mathematik'*, op. cit.

> *nicht geachtet werden soll, an denen somit alles, was wir an ihnen unterscheiden, bestimmt ist, sobald nur ihre Theile bestimmt sind, verdienen es eben um dieser Beschaffenheit willen, mit einem eigenen Nahmen bezeichnet zu werden. In Ermangelung eines andern tauglichen Wortes erlaube ich mir das Wort »Menge« zu diesem Zwecke zu brauchen."*
>
> [Now, *'Inbegriffe',* where no attention is to be paid to the way in which their parts are connected with one another, and where therefore everything we distinguish about them is determined, as soon as only their parts are determined, deserve to be called by a name of their own precisely for the sake of this nature. In the absence of another suitable word, I will take the liberty of using the word *'set'* (*'Menge'*) for this purpose.]
>
> B. BOLZANO: *'Größenlehre'*, Gesamtausgabe, Volume II, A,7; p. 152.

The German word *'Menge'* is not unsuitable here, because, in colloquial German language, the verb *'mengen'* has the meaning of *'bringing together', 'mixing', 'stirring into each other', 'connecting in a loose way',* etc. In a *'Menge',* therefore, different kinds of substances are brought together in a disorderly manner. The Old High German root of the word *'Menge'* is *'managi',* and this is the abstract formation of *'manac'* (New High German: *'manch'*).

The English translation *'set'*, however, speaks to us more of a group or collection of objects that belong together because they are similar or complementary to each other.

BOLZANO thus distinguishes between *'Inbegriffe'* ('structured sets') and *'Mengen'* ('sets'). As an example, he mentions, in his *'Größenlehre'* (pp. 102-103), the *"Inbegriff aller in einem Staate lebenden Menschen"* ["*the structured set of all people living in a state"*]. For BOLZANO, this is a hierarchically-ordered totality in which first the king/the queen, then his/her ministers, etc., are highlighted. According to BOLZANO, the *'structure of sovereign power'* should be taken into account in this example of a totality. It is, hence, an *'Inbegriff'* (an embodiment of a structure). A second example is the *'Inbegriff'* of all points that lie on a straight line between two given points *A* and *B*. Here, the continuous linear arrangement of the points is included (*'inbegriffen'*) in the concept (*Begriff*) of a straight line (cf. A.G. KÄSTNER *'Anfangsgründe der Arithmetik, Geometrie etc.',* Göttingen 1786, p. 3 and p. 165 ff, as well as BOLZANO *'Größenlehre'* p. 189). Another good example is the *'Inbegriff der rellen Zahlen'*, in which the totality of all real numbers is intended, but with its algebraic and topological structure *included* (*'inbegriffen'*). – Notice that this old-fashioned terminology was replaced in the 20th century by references to so-called *'structures'*.

As an example of a *'Menge'* (*'set'*), BOLZANO mentions a pile of money where the value does not depend on the order (or disorder) of the pile.

In contrast to the view of the mathematicians and philosophers of antiquity, BOLZANO was of the opinion that *'Mengen'* (*'sets'*), since they are *'quantities'*, are to be regarded as *things*, and indeed as unchangeably existing things. For BOLZANO, *'Mengen'* (*'sets'*) are the *being together*, or the *gathering*, of certain objects, and this *mere being together* is itself an object.

The ontological status of *'Inbegriffe'* and *'Mengen'* has thus changed. BOLZANO wrote about this in the *'Einleitung zur Größenlehre'* (Gesamtausgabe, Volume II, A,7, pp. 100-101):

> *"Ich behaupte also, daß Inbegriffe bestehen nicht dadurch, daß wir sie denken, sondern umgekehrt, daß wir nur dann sie mit Wahrheit denken können, wenn sie bestehen auch ohne daß wir sie denken."*
>
> [Hence, I claim that embodiments (*Inbegriffe*) exist not through our thinking, but conversely, that we are able to think of them only if they also exist without our thinking of them.]
>
> B. BOLZANO: *'Gesamtausgabe'*, Volume II, A, 7; p. 101.

BOLZANO has thus taken a position comparable to the position of the 'realists' in the medieval dispute over universals. But *realism concerning sets* and *realism concerning universals* are different positions, because universals are *intensionally* determined objects, while sets are *extensionally* determined.

BOLZANO also stressed that *'Mengen' ('sets')* need not be definable. Thus, they do not have to be *extensions of concepts*, they do not have to be thought of, and they do not have to be given by an explicit enumeration of all their elements. In his booklet on the *'Paradoxien des Unendlichen'*, §14, p. 17, BOLZANO wrote:

> *"Es gibt also Mengen ..., auch ohne daß ein Wesen, welches sie denkt, da ist."*
>
> [So sets exist ... even if there is no one who thinks of them.]

It is noteworthy that the human *soul* has no role in the formation of 'sets'. 'Sets' can be *recognized* and *thought of,* but they are not created by the soul (the anima rationalis) (cf. Chapter 13). An interference of psychology into set theory is therefore rejected by BOLZANO. According to BOLZANO, *sets* belong to the world of pre-existing things, the world of material or of spiritual things.

From then on, other mathematicians also consciously and explicitly emphasized the thing-like nature of sets. RICHARD DEDEKIND wrote, in his famous booklet *'Was sind und was sollen die Zahlen?* (1888), right in the first paragraph:

> *"Ein solches System S (oder ein Inbegriff, eine Mannigfaltigkeit, eine Gesamtheit) ist als Gegenstand unseres Denkens ebenfalls ein Ding."*
>
> [Such a System S (or an embodiment, a manyfold, a totality) is as an object of our thinking also a thing.]

FELIX HAUSDORFF wrote, in his *'Grundzüge der Mengenlehre'* (Leipzig, 1914), right on the first page:

> *"Eine Menge ist eine Zusammenfassung von Dingen zu einem Ganzen, d.h. zu einem neuen Ding."*
>
> [A set is a comprehension of things into a whole, that is, into a new thing.]

Similar formulations are also to be found in FREGE's work (see, for example, his letter of April 26, 1925, to RICHARD HÖNIGSWALD, in FREGE*'s 'Wissenschaftlicher Brief-wechsel',* op. cit., p. 85) and in the work of many other authors.

BOLZANO (just like KÄSTNER) always spoke of the *'parts'* of a *'set'* or of an *'Inbegriff'*. He never spoke of *'elements'*. Therefore, it is sometimes unclear whether he meant elements ($\in$) or subsets ($\subseteq$) when he spoke of 'parts'. But it is sufficiently clear that, in all the relevant places here, the expression *'part of a set'* always means what we (since R.

DEDEKIND, 1872, G. CANTOR, 1878) call an *'element of a set'*.[3] For BOLZANO, for example, the objects that fall under a concept $B(x)$ are the *parts* of the extension of $B(x)$, i.e., in modern parlance, are the *elements* of the 'extension of the concept $B(x)$' (cf. BOLZANO *'Wissenschaftslehre',* Volume 1, p. 298).

Since the arrangement of the parts (elements) of a *set* is irrelevant, the following is obviously true:

> *"Die Theile, aus denen eine Menge besteht, bestimmen sie, und zwar vollständig und alle auf einerley Art."*
>
> [The parts out of which a *set* consists determine the set completely and all in a uniform manner.]
>
> B. BOLZANO: *'Größenlehre'*, complete edition, volume II, A, 7, p. 152.

This statement is now called the *'axiom of decisiveness'* or *'axiom of extensionality'*. Sets are thus determined by their extension, i.e., by their elements alone. Notice that, in contrast, neither concepts ('Begriffe') nor 'Inbegriffe' (embodiments, structured sets) are generally determined by their extension! But notice also that this principle was formulated by BOLZANO in a work that was published only posthumously, no earlier than 1975. It was not known then, neither to CANTOR nor to DEDEKIND nor to any other person of that time. Independently of BOLZANO, it was also formulated by DEDEKIND, in his booklet *"Was sind und was sollen die Zahlen?"* (1888, §1, art. 2). ERNST ZERMELO also formulated this principle in his system of axioms for set theory in 1908. Presumably, ZERMELO took this axiom from DEDEKIND, calling it *'Axiom der Bestimmtheit'* ('axiom of decisiveness').

The word 'set' is now much more than a mere façon de parler. Sets are now understood to have the attribute of objecthood and may, hence, occur as elements in other sets. This was emphasized explicitly by BOLZANO:

> *"Die Theile, aus denen ein Inbegriff bestehet, können selbst wieder Inbegriffe seyn."*
>
> [The elements which constitute a set can themselves be sets.]
>
> B. BOLZANO: *'Größenlehre',* complete edition, vol. IIA7, p. 102.

The exhaustion of this possibility to construct new sets from given sets led to the enormous theory of transfinite sets as we have known it since GEORG CANTOR, ERNST ZERMELO, FELIX HAUSDORFF, WACLAW SIERPINSKI, AZRIEL LÉVY, RONALD JENSEN, ROBERT SOLOVAY and many others.[4]

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[3] The objects that make up a *set* have been called *'elements'* by DEDEKIND and CANTOR. DEDEKIND used this terminology as early as 1872, in the first draft of his booklet *'Was sind und was sollen die Zahlen?'* (published in PIERRE DUGAC, op. cit., pp. 293-309). CANTOR, however, used it only from 1878 on (cf. 'CANTOR*s Gesammelte Abhandlungen'*, op. cit., pp. 119-133). The Latin word *elementum* denotes those units out of which something else is composed, such as a word is composed out of letters (cf. footnote 1 in Chapter 4). If a is an element of the set M, then, according to the Italian logician GIUSEPPE PEANO (1889), we write $a \in M$. The character $\in$ is a stylized ε and stands for the Greek *esti* (ἐστί = is).

[4] Notice that, in the *intuitionism* of L.E.J. BROUWER (1881-1966), a *set* is understood only as a linguistically formulated law (cf. BROUWER in Math. Annalen 93 (1925), pp. 244-258). For BROUWER, *sets* are not 'things', and therefore cannot become elements of new sets in his theory. For example, the set of all real numbers can no longer be formed in BROUWER*'s* intuitionism.

15.3 Cantorian set theory

BOLZANO began the development of set theory. Only about 30 or 40 years later, GEORG CANTOR went beyond BOLZANO, indeed, far beyond, with his proofs (1873, 1890) that there are infinite sets of different sizes, with his *Theories of infinite ordinal and cardinal numbers* (*Alephs*: $\aleph_0, \aleph_1, \aleph_2, ..., \aleph_\omega, \aleph_{\omega+1}, ...$) and with his (later so-called) *Descriptive Set Theory*.

From a philosophical point of view, it is important to note that CANTOR was able to distinguish between the *transfinite* and the *absolute-infinite*. Before we can go into this, we need to discuss CANTOR*'s* discovery that, among infinite sets, there are sets of different sizes.

One of CANTOR*'s* most spectacular results was the proof of the uncountability of the set $\mathbb{R}$ of all real numbers (1873/1874). CANTOR published a second proof in 1890. We shall reproduce and analyse this elegant second proof. [In the following , we will use $\{x; \Phi(x)\}$ - as usual - to denote the *set, multitude, totality* or *class* of all objects x that have the property Φ, i.e., the extension of the concept Φ].

Theorem 1 (G. CANTOR, 1873/1874) *There are more real numbers than any given countable set of real numbers.*

Proof (CANTOR, 1890). Let $r_1, r_2, r_3, ...$ be any given countable sequence of real numbers, and, without loss of generality, assume that they all lie in the interval between 0 and 1. Every number r_n in this sequence has exactly one non-terminating decimal expansion: $r_n = 0,r_{n1}r_{n2}r_{n3}...r_{nk}.....$ We define a new number $s = 0,s_1s_2s_3 ... s_k ....$ by putting $s_k = r_{kk}+1$ if $r_{kk} \leq 5$, and $s_k = r_{kk} - 1$ otherwise. The given decimal expansion of s apparently does not terminate after finitely many steps. Also, s is in the interval between 0 and 1, but is different from all numbers r_n , since s differs from r_n at the n^{th} position after the decimal point. Thus, a 'new' number s is found, Q.E.D.

Let us make two comments about the above proof.

(1) The proof uses a so-called *diagonal argument,*[5] because the numeral r_{kk} , whose index

[5] With a similar *diagonal argument,* the German Mathematician PAUL DU BOIS-REYMOND (op. cit. 1872/1873, p. 89) had already shown earlier, that, in the set of all monotonously growing divergent real functions *f*: $\mathbb{R}^+ \to \mathbb{R}^+$, which are graduated according to their final course, for every countably-infinite ascending sequence of functions, there is always a further function dominating all of them (cf. U. FELGNER 2002, op. cit., p. 650). The *method of diagonalization* is hence due to PAUL DU BOIS-REYMOND, and not to CANTOR. But notice that, although DU BOIS-REYMOND*'s* proof would immediately (!) imply that the totality of all real functions is uncountable, he did not arrive at that result. He had neither the concept of a totality of things as an existing thing, nor the concept of uncountability. However, the first *Cantorian* proof of the uncountability of the set of all real numbers (of December 7, 1873) is also based on a diagonal argument, even if it is hidden and as such possibly remained unnoticed by CANTOR. The proof is based on CANTOR's own theory of real numbers, which he published in March 1872 (Ges. Abhandl., pp. 92-94). In that theory, each real number is an equivalence class of CAUCHY-

belongs to the *'diagonal'* $\{(k,k);\ k \in \mathbb{N}\}$, was replaced by a different numeral s_k .

(2) In the formulation of theorem 1, we have imitated the formulation of EUCLID's prime number theorem in order to emphasize that the given proof - strictly speaking - only shows that the multitude of all real numbers is *potentially uncountable* (cf. our discussion of the proposition in Chapter 5, section 5.5). CANTOR tacitly assumed that the totality $\mathbb{R}$ of all real numbers can be considered as a well-defined *actual infinite* mathematical object. Under this assumption, the given proof shows that $\mathbb{R}$ is an *'uncountable set'*. Using CANTOR's theory of infinite cardinal numbers, which is based on HUME's definition[6] of the equality of numbers as the equipotency of their underlying sets, then theorem 1 can also be pronounced as follows: *The set* $\mathbb{R}$ of *all real numbers has a greater cardinality than the set* $\mathbb{N}$ *of all natural numbers.*

In 1890, CANTOR was able to generalize the above-mentioned theorem considerably by showing that the power set $\mathsf{P}(M) = \{X;\ X \subseteq M\}$ of any set M always has greater power than the set M itself. If we use $|M|$ to denote the power (or cardinal number) of the set M, then we always have $|M| < |\mathsf{P}(M)|$.

ERNST ZERMELO presented this theorem ten years later in his lectures on set theory during the winter semester of 1900/1901 in Göttingen. But he modified the proof a little bit, and thereby found the following simple, but important, theorem (see U. FELGNER 2012, op. cit., and E. ZERMELO*: 'Gesammelte Werke'*, vol. 1, p. 167):

Theorem 2 (ZERMELO*, 1900/1901*): *Every set M has at least one subset A that is not an element of M*: $A \subseteq M \ \&\ A \notin M$.

Proof: Let A be the set of all of those elements of M that do not contain themselves as an element, $A = \{x;\ x \in M \ \&\ x \notin x\}$. Then, we have $A \notin M$, because, otherwise (i.e., assuming $A \in M$), $A \in A \Leftrightarrow A \notin A$ would immediately follow, which would be contradictory, Q.E.D.

ZERMELO also included this theorem as theorem 10 in his *'Untersuchungen über die Grundlagen der Mengenlehre'* (Math. Ann. 65 (1908), pp. 261-281). Also, the proof of theorem 2 is based on a diagonal argument, because, in the definition of set A, the formula $x \notin x$ was used.

From theorem 2, the following **corollary** is immediately obtained: *There are more sets than any given set of sets*. In the formulation of this corollary, we have again imitated the formulation of the prime number theorem of EUCLID.

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sequences in the field of rational numbers, $\mathbb{Q}$. In order to prove that the intersection of a (nested) decreasing sequence of closed intervals in $\mathbb{R}$ is non-empty, a suitable CAUCHY-sequence (in $\mathbb{Q}$) must be constructed by diagonalzation out of the members of the endpoints of the intervals.

[6] HUME's principle reads as follows: *"When two numbers are so combined, as that the one has always an unite answering to every unite of the other, we pronounce them equal.*" This principle was introduced in HUME's *'Treatise of Human Nature'*, I,3,1 (1739), op. cit. A German translation of this book appeared in Halle 1790. HUME's Principle was also presented in full detail in JOH. JULIUS BAUMANN*: 'Die Lehren von Raum, Zeit und Mathematik in der neueren Philosophie, nach ihrem ganzen Einfluss dargestellt und beurteilt'*(Berlin 1868, op. cit.). We can assume that either HUME's *Treatise* or BAUMANN's work was known to CANTOR.

15.4 The occurrence of set-theoretical antinomies

From ZERMELO's theorem 2, it also follows *that the totality* (or *multitude*) *V of all sets cannot be a set*, because, otherwise, there would have to be a set that is not in *V,* although *V*, by definition, contains all sets. The assumption that the universe *V* of all sets is a set thus leads to a contradiction.

BOLZANO already suspected that dealing with *unlimited* quantities could lead to contradictions. He wrote, in 1831, in his *'Mathematisches Tagebuch'*, issue 22, page 1968 (see also BOLZANO*'s 'Philosophische Tagebücher 1827-1844',* Gesamtausgabe, Reihe II-B, Band 18/2, p. 29):

> *"das All der A, wo A eine Vorstellung von beschränktem Umfange bezeichnet, z.B. Mensch – läßt sich recht wohl denken. Sobald man aber für A die weiteste aller Vorstellungen, nämlich die eines Etwas überhaupt setzt, so entsteht die Schwierigkeit, daß die Vorstellung »das All der Gegenstände« oder »das All von Allem« oder »das absolute All«, eigentlich auch sich selbst, weil ja dieses All auch wieder »Etwas« ist, umfassen sollte; welches doch ungereimt ist. Ich glaube deshalb, daß man diese Vorstellung in der Tat zu den widersprechenden (imaginären) zählen müsse, gerade wie die Vorstellung von der geschwindesten Bewegung. – Aber auch schon das All der Inbegriffe wäre eine solche sich selbst widersprechende Vorstellung."*
>
> [the *All* of *A*, where *A* denotes a concept of limited extension, e.g. human being - can be thought of quite well. But as soon as one takes for *A* the broadest of all conceptions, namely, that of something, then a difficulty arises in that the conception of »the all of objects« or »the all of everything« or »the absolute all«, will contain itself, because this »all« is also »something« again; which, after all, is unrhymed (absurd). I therefore believe that this concept must indeed be counted among the contradictory (imaginary) ones, just like the idea of the fastest movement. - But already the totality of all embodiments would be such a self-contradictory conception.]
>
> Quoted after JAN SEBESTIK: *'La classe universelle et auto-appartenance chez Bernard Bolzano'*. Eleutheria (ELEYΘERIA), Athens 1986, pp. 72-84.

BOLZANO probably calmed down very soon afterwards, and wrote in his diary:

> *"Responsio. Nicht doch! Diese Begriffe sind nicht widersprechend."*
> [Answer: Not at all! These concepts are not contradictory.]

Unfortunately, he did not pursue the antinomy of the universe, i.e., the "set of all sets", of which he makes his suspicions known here.

GEORGE BOOLE (1815-1864) had also come across the 'universal set' (*"class 1",* as he called it), *"which comprehends every conceivable class of objects",* but he did not recognise its antinomic character (cf. G. BOOLE: *'The Mathematical Analysis of Logic',* Cambridge 1847, p. 15). The same applies to R. DEDEKIND, who, in his booklet *'Was sind und was sollen die Zahlen?* (Braunschweig, 1888, p. 14, Theorem 66), considered *the totality of all things that might happen to be objects of his thinking* ("*die Gesamtheit S aller Dinge, welche Gegenstand (seines) Denkens sein können*.")

CANTOR, on the other hand, had (around 1883) very clearly recognized the antinomic character of the 'totality of all sets' (i.e., the universe *V* of all sets) - see the treatise *'Georg*

Cantor und die Antinomien der Mengenlehre' by WALTER PURKERT, 1986, op. cit. That the universe *V* of all sets cannot be a *set*, CANTOR (probably) proved as follows.

If *V* were a set, then the totality of all subsets of *V*, called $\mathsf{P}(V)$, would also be a set. It would follow $\mathsf{P}(V) \subseteq V$, since *V* is the universe of *all* sets. Consequently, the cardinality of $\mathsf{P}(V)$ would be less than or equal to the cardinality of *V*. The power set of any set, however, as CANTOR was able to prove in 1890 (see above), really has greater cardinality than the set itself - a contradiction!

Thus, as proved by CANTOR (ca. 1883/1890) and by ZERMELO (ca. 1900*),* we know that *the totality V of all sets cannot be a set itself*. The extension of the concept of a set (in the sense of ARNAULD-NICOLE, but in the version that sets are to be regarded as things) cannot itself be a *set,* because, otherwise, contradictions would arise.

In 1897, CESARE BURALI-FORTI (1861-1931) published an essay (op. cit.) in which he considered the well-ordered system Ω of all ordinal numbers and then moved on to the order type $\Omega+1$ by adding another element. Obviously, we have $\Omega < \Omega+1$, because $\Omega+1$ is the successor of Ω. But $\Omega+1 < \Omega$ is also true, because the order type Ω of *all* ordinal numbers is strictly greater than each individual ordinal number. It follows that $\Omega < \Omega$, which is contradictory.

What does this result mean? Does it mean that CANTOR's set theory is contradictory, or does it just mean that Ω is not a set either?

BURALI-FORTI had submitted his paper for publication in February 1897. It was accepted for printing and appeared in the *Rendiconti del Circolo Matematico di Palermo* in the same year. In August 1897, BURALI-FORTI and CANTOR met at the First International Congress of Mathematicians in Zürich. One may assume that they talked to each other, although nothing has been officially recorded about their meeting. It is only known that BURALI-FORTI*'s* result did not worry CANTOR at all. CANTOR (presumably) thought that the multitude (or totality) Ω of all ordinal numbers is no more a 'set' than the multitude *V* of all sets, that both Ω and *V*, as *absolute-infinite multitudes*, are not real 'things' and that, therefore, $\Omega+1$ cannot be formed and, in this respect, the proof of the assertion $\Omega < \Omega$ is erroneous (cf. CANTOR*'s* 'Ges. Abhandl.', op. cit, p. 205, lines 25-26, and CANTOR*'s* letter of August 3, 1899, to DEDEKIND, in *'G. Cantor's Letters',* op. cit., pp. 407-411).

CANTOR was thus of the opinion that BURALI-FORTI had misunderstood the term *'Menge'* (*'set'*) and had also used it incorrectly. But many mathematicians who learned of BURALI-FORTI*'s* paradox were confused and worried. They could not find an error in his argumentation, and even considered the BURALI-FORTI *paradox* to be a veritable *antinomy*.[7] So why shouldn't Ω be a set? When is a multitude of things a 'set' and when not?

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[7] A *paradox* is a statement that, at first glance, seems to be false, but nevertheless is true. *Para* (παρά) means 'next to it', 'on the side of', 'against' and *doxa* (δόξα) is 'the opinion'. A *paradox* is thus a statement that contradicts a widespread opinion.
An *antinomy* (ἀντινομία), on the other hand, is a 'contradiction of the law with itself'. The prefix *anti* (ἀντί) means 'instead of', 'against', 'contrary' and *nomos* (νόμος) is '*the law valid for all'.* In logic, an *antinomy is* a statement that contradicts itself. EUBULIDES, who lived around 300 B.C.E., formulated a famous antinomy: 'If I say *»I am lying«,* is this statement itself a lie or a true statement?' (see DIOGENES LAËRTIOS in his biography of the philosopher EUCLEIDES of Megara).

In 1903, the book *'The Principles of Mathematics'* by BERTRAND RUSSELL (1872-1970) was published, in which RUSSELL reported, in Chapter X, on the antinomy of the set of all sets that do not contain themselves as an element.[8] This antinomy soon became the most famous antinomy in set theory, partly because RUSSELL illustrated it (though not until 1918, op. cit., p. 228) with the popular example of the *barber who shaves all the inhabitants of his home town who do not shave themselves. Does he shave himself?* RUSSELL reported that he found the aforementioned set-theoretical antinomy in the summer of 1902, or perhaps even in 1901 (cf. chapter 14). ZERMELO had already come across this antinomic 'set' in the winter of 1900/1901 (or earlier).[9] It has therefore become common practice to speak of

ZERMELO-RUSSELL'*s antinomy*:

if $Z = \{x \in V; x \notin x\}$ *were to be a 'set', then* $Z \in Z \Leftrightarrow Z \notin Z$ *would be true.*

The *Zermelo-Russellian* class *Z* therefore cannot be considered as a ‘set’, and not as a ‘thing’ either, if contradictions are to be avoided. But should that be the only reason to dismiss *Z* from being a ‘set’? *Z* is an extension of a concept, a collection, a multitude (or totality) of sets and, hence, could be a *‘set’* just like the infinite multitude of all natural numbers $\mathbb{N}$. Why should $\mathsf{P}(M)$ always be a *‘set’* for every *‘set’ M*? How is the term ‘set’ defined? - Many mathematicians asked themselves these and similar questions at the turn of the century. CANTOR was asked these questions probably by several colleges as well.

CANTOR was alarmed and, from 1897 on, he tried to clarify his point of view in letters to DEDEKIND, HILBERT, JOURDAIN, SCHÖNFLIES and CHISHOLM-YOUNG. CANTOR had believed, until then, that he could define the concept of a set in such a way that the familiar collections $\mathbb{N}$, $\mathbb{R}$, $\mathsf{P}(\mathbb{R})$, $\mathsf{P}(\mathsf{P}(\mathbb{R}))$, etc., fell under it, but the unbelievably large multitudes, such as *V, Ω* and *Z*, did not. In the following, we will discuss his attempts at a definition.

15.5 The Cantorian concept of a ‘set’ (‘Menge’)

CANTOR, after having discovered that there are infinite sets of different sizes, thought long and thoroughly about the concept of infinity and studied much that had been written about this concept since antiquity. He found that different concepts of infinity had been handed down to us from philosophy and theology. For some of them (ARISTOTLE and others),

'not being finite'

And, for the others (JOHANNES PHILOPONOS, BONAVENTURA, NICOLAUS CUSANUS, BENEDICTUS DE SPINOZA,...),

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[8] This book may confuse its readers, since, in Chapter X, entitled *"The Contradiction"*, RUSSELL presents the antinomy of the class of all classes that do not contain themselves as an element, and shows that this antinomy is derivable in FREGE's theory of natural numbers. But, in Chapter XI, RUSSELL presents his own theory of cardinal numbers, which is inconsistent precisely for the same reason. It is quite surprising that RUSSELL did not realize this *"absolutely fatal formal defect"* (RUSSELL, #110, p.114) in his own theory. For more details, see Chapter 14, section 14.3.

[9] See also ZERMELO *'Gesammelte Werke'*, op. cit., vol. 1, pp. 167-168, and B. RANG - W. THOMAS, op. cit.

'not being surpassable in size'

had been characteristic properties of the infinite.[10] CANTOR took up this distinction and called the infinite multitudes, which cannot be exceeded in size, *'absolute-infinite'*. The infinite multitudes, which lie between the finite and the absolute-infinite ones, have in common with the finite multitudes that they can be exceeded in size. CANTOR called them *'transfinite'* multitudes (cf. CANTOR*'s 'Ges. Abhandl.'*, pp. 175, 205, 385).

In this terminology, the familiar objects $\mathbb{N}$, $\mathbb{R}$, $\mathsf{P}(\mathbb{R})$, $\mathsf{P}(\mathsf{P}(\mathbb{R}))$, etc., would be *transfinite* and the incredibly large multitudes, such as *V, Ω* and *Z, absolutely infinite* (cf. CANTOR*'s* letter of 05.10.1883 to WILHELM WUNDT).

CANTOR spoke carefully of the *absolute,*[11] since this term has a long tradition in philosophy and theology. In WALCH*'s 'Philosophisches Lexicon'* (Leipzig 1775, 1st part, p. 33), it says, for example:

> *"Insgesamt nennt man dasjenige ein Absolutum, was keinen Respect oder keine Beziehung gegen eine andere Sache hat."*
>
> [In general one calls that an *Absolutum* what has no respect or relationship against another thing.]

The adjective *'absolute'* is therefore used to denote that which exceeds a given scale of norms and for which a comparison is no longer possible. In contrast to the transfinite, *'the absolute infinite eludes mathematical determination'* (G. CANTOR*, 'Ges. Abhandlungen'*, p. 405). In a letter of June 20, 1908, to GRACE CHISHOLM-YOUNG, CANTOR wrote:

> *"Was über dem Finiten und Transfiniten liegt, ist ... das »Absolute«, für den menschlichen Verstand Unfassbare, also der Mathematik gar nicht unterworfene, (das) Unmessbare.*"[12]
>
> [What lies above the finite and transfinite is ... the »absolute«, that which is incomprehensible to the human mind, and hence not subjected to mathematics, it is (the) immeasurable.]

CANTOR was guided by theological convictions in the separation of the Absolute Infinite from the Transfinite (cf. M. HALLETT, 1984, op. cit.).

The *Absolute Infinite*, according to CANTOR, is not realized in the world of created things, material or spiritual. It is, as CANTOR wrote in a letter of January 22, 1886, to Cardinal J.B. FRANZELIN in Rome,

> "*ein auf ewig unerschaffenes Infinitum",* ein *"Infinitum aeternum increatum ... das sich auf Gott und seine Attribute bezieht."*
>
> [an infinitum that is eternally uncreated, an *Infinitum aeternum increatum* ... that refers to God and His attributes.]

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[10] *"Infinito autem nihil maius*" - Quoted from JOHANNES PHILOPONOS *'Commentary on the first three books of the Aristotelian lecture on physics'*, BONAVENTURA *'Commentary on the Sentences of Peter Lombardus'*, II, dist.1, pars 1, art.1, Q.2, and BENEDICT SPINOZA *'Ethics'*, Part 1, Def. 2.

[11] In Latin, 'absolutus*'* means *"detached, not in need of any further definition, unconditional, not dependent on anything else"*. The verb *'solvere'* means "to loosen, to lift up, to detach".

[12] These statements of CANTOR*'s* are reminiscent of DESCARTES' saying that we humans could not comprehend the infinite: *"Nostre ame, estant finie, ne peut comprendre l'infiny"* (see Chapter 6).

In contrast, the *"infinitum creatum sive transfinitum"* [*'the created infinite or transfinite'*] can be stated everywhere *'where in the natura creata an actual infinite must be ascertained'* (see CANTOR*'s* '*Letters*', op. cit., pp. 254-257, or CANTOR*'s* '*Ges. Abhandlungen*', op. cit.).

In his *'Mitteilungen zur Lehre vom Transfiniten'* (1887, Ges. Abhandl., p. 378 ff), CANTOR once again explained this distinction as follows:

> *"Es wurde das Aktual-Unendliche nach drei Beziehungen unterschieden: erstens sofern es in der höchsten Vollkommenheit, im völlig unabhängigen, außerweltlichen Sein, in Deo realisiert ist, wo ich es Absolutunendliches oder kurzweg Absolutes nenne; zweitens sofern es in der abhängigen, kreatürlichen Welt vertreten ist; drittens sofern es als mathematische Größe, Zahl oder Ordnungstypus vom Denken in abstracto aufgefaßt werden kann. In den beiden letzten Beziehungen, wo es offenbar als beschränktes, noch weiterer Vermehrung fähiges und insofern dem Endlichen verwandtes Aktual-Unendliches sich darstellt, nenne ich es Transfinitum und setze es dem Absoluten strengstens entgegen."*
>
> [The Actual-Infinite has been distinguished according to three relationships: firstly, insofar as it is realized in God, in the highest perfection, in the completely independent, extra-worldly being, where I call it the Absolute-Infinite or, in short, the Absolute; secondly, insofar as it is represented in the dependent, creaturely world; thirdly, insofar as it can be understood as a mathematical quantity, number or ordertype conceived in abstracto from thinking. In the last two relationships, where it appears to be a limited actual infinity capable of further enlargement and thus related to the finite, I call it transfinite and oppose it to the Absolute most strictly.]

These persuasions led CANTOR to consider finite and transfinite multitudes as *things* and, at the same time, as *quantities* (in the traditional sense, cf. EULER*'s 'Vollständige Anleitung zur Algebra'*, 1770), and therefore also to call them *sets*. The absolute-infinite multitudes, however, are not to be understood as *things,* not even as *quantities*, and therefore not as *sets.*[13]

CANTOR*'s* distinction between *transfinite* and *absolute-infinite* multiplicities is reminiscent of the medieval dispute over whether multiplicities are *res* [things] or only *voces* [names], that is, whether they have substantial existence or are merely linguistic designations. For CANTOR, the transfinite multiplicities are *res*, but the absolute-infinite multiplicities are *voces*, since they do not have an inner-worldly existence. The finite and the transfinite multiplicities, i.e., the *sets*, possess (inner-worldly) reality, and can thus be

[13] Unfortunately, CANTOR occasionally forgot this idea and, in his publications, he also spoke of the *'set'* of all cardinal numbers, as well as of the *'set'* of all ordinal numbers (cf. CANTOR*'s 'Gesammelte Abhandlungen'*, op. cit., pp. 280, 295, 321, 419 and 447). This irritated many readers of *Cantor's* treatises - including the author of this book in his earlier publications - and led them to regard CANTOR's concept of sets as a contradictory concept. CANTOR, however, has explicitly pointed out, in many other places, that, for example, *the 'totality of all Alephs ...* is *an absolutely infinite totality'*, which can be *clearly distinguished* from the *'transfinite sets'* (as stated in CANTOR*'s* letter of September 26, 1897, to HILBERT). WALTER PURKERT has convincingly demonstrated, in his paper (1986, op. cit.), that CANTOR never seriously conceded the predicate of being a set to absolute-infinite totalities and that, in this respect, CANTOR*'s* concept of a set does not give rise to the known antinomies.

thought of without contradiction. They are available to humans in their thinking as *'finished'* (*"fertige"*) created objects.

CANTOR, although he was guided by ideological and theological convictions, tried to explain his view of the concept of a set in a way that was free of any philosophical position. He first tried to give a definition of the concept of a set in 1883. He wrote:

> *"Unter einer »Mannigfaltigkeit« oder »Menge« verstehen ich ... allgemein jedes Viele, welches sich als Eines denken läßt, d.h. jeden Inbegriff bestimmter Elemente, welcher durch ein Gesetz zu einem Ganzen verbunden werden kann."*
>
> [By a 'manifold' or 'set' I understand in general that Many which can be thought of as One, i.e. every collection of certain elements which can be bound up into a whole through a law.]
>
> G. CANTOR *'Gesammelte Abhandlungen'*, Berlin 1932, p. 204.

In this first attempt at a definition of the concept of a *'set'*, definability and conceivability are still assumed. (For CANTOR, 'sets' and 'embodiments' are the same.) In a modified form, CANTOR gave his definition, in 1895, in a much more general way as follows:

> *"Unter einer »Menge« verstehen wir jede Zusammenfassung M von bestimmten wohlunterschiedenen Objekten m unserer Anschauung oder unseres Denkens (welche die »Elemente« von M genannt werden) zu einem Ganzen."*
>
> [By a "set" we understand any combination *M* of certain well-differentiated objects of our intuition or our thinking (which are called the "elements" of *M*) into a whole.]
>
> G. CANTOR: *'Gesammelte Abhandlungen'*, Berlin 1932, p. 282.

CANTOR placed this definition in front of his two-part treatise *'Beiträge zur Begründung der transfiniten Mengenlehre'* (1895, 1897), his last publication in mathematics, and exhausted its contents in a long, impressive series of beautiful and profound theorems. CANTOR was convinced that he did not need to place axioms or postulates at the beginnings of his theory, but only a definition of the basic concept. He believed that his set theory described a spiritual incorporeal world given to us, and that, in this respect, all the theorems he proved were truths of apodictic certainty.

The definition that CANTOR used as a basis for his set theory seems, at first sight, to be simple and immediately understandable. It even became the much-cited classical definition of the concept of a set. But it is, if read carefully, word for word, rather dark. For example, what is a *'combination (or gathering) into a whole'* and when can a multitude of things be *'thought as one'* and when not? Are the *'absolute-infinite totalities'* (like *V*, *Ω* and *Z*), of which CANTOR spoke in a letter (dated September 26, 1897) to HILBERT, not also 'totalities' of objects that are *combined into a whole*?

FREGE considered CANTOR's 'definition' to be *'unclear'* (cf. FREGE: *'Kleine Schriften'*, op. cit., p. 164). DEDEKIND asked CANTOR (in his letter of August 29, 1899) the question as to what he meant *by 'the being together of all elements of a multitude'* (cf. P. DUGAC, *'R. Dedekind et les fondements des Math.'*, 1976, p. 261). In his monograph *'Grundzüge der Mengenlehre'* (Leipzig 1914, p. 1), FELIX HAUSDORFF even refused to accept CANTOR's 'definition' as a definition.

Nevertheless, CANTOR's 'definition' contains very clear indications of the intended content of his concept of a 'set'. That an infinite multitude is a *set* when it *can* be *'thought of as one',* or when *'its elements are bound up to form a whole'*, or are *'combined to form a whole',* as CANTOR tried to explain in numerous letters, is to indicate that these are things (!) that belong to the created world and are, hence, *finite* or *transfinite*, but not *absolute-infinite*. With a multitude A, which can be regarded as a *whole*, it must (in principle) be clear what its elements are and what they are not: it cannot remain undecided whether, for example, $A \in A$ or $A \notin A$ is true. - It must be admitted, however, that CANTOR did not succeed in giving a satisfactory and precise definition of the concept of a set.

Soon after, the question arose as to whether it is possible to give an *explicit definition of the essence* (see Chapter 3) of a set. However, if the concept of a 'set' is an '*irreducible*' concept, which cannot be defined with even simpler concepts, then no explicit definition can be given. But, then, it is still possible to introduce the set concept in an axiomatic way. Such an introduction was carried out by ZERMELO in 1908. His approach was expanded and put into a final form in the following years.

15.6 An implicit definition of the concept of a 'set' (ZERMELO, QUINE, et al.)

Since it does not seem possible to define the mathematical concept of a 'set' *explicitly*, one must probably be satisfied with an *implicit* definition [14]. The concept of a *'set'* is then defined only in that one writes down a list of statements in which the word *'set'* occurs, *but without presupposing that a certain content is associated with this word!* In these statements – the *'axioms of set theory'* – it is postulated that certain objects exist and that they are called *'sets'*, and that further *'sets'* can be obtained by applying the operations mentioned in the axioms. In this way, the concept of a *'set'* received as much content as results from the operative handling of the axioms.

ERNST ZERMELO proposed such a system of set-theoretical axioms in 1908. In the following 50 years, THORALF SKOLEM (1887-1963), ADOLF ABRAHAM FRAENKEL (1891-1965), JOHANN VON NEUMANN (1903-1957), PAUL BERNAYS (1888-1977) and WILLARD VAN ORMAN QUINE (1908-2000), in particular, worked on a more precise definition, which we will not present here in full detail. The resulting axiom system could be called ZSFNBQ, but is usually called ZSF (or, quite often, just ZF). It consists (in simplified, colloquial formulation) of the following axioms:

(I) Extensionality Axiom : *Two sets are identical when they have the same elements.*
(II) Infinity Axiom: *There is an infinite set.*
(III) Axiom of union: *For every set A, there exists a set whose elements are the elements of the elements of A.*
(IV) Axiom of power sets: *For every set A, there exists a set whose elements are the subsets of A.*
(V) Replacement Axiom: *If the elements of a set are replaced by some other things, then the resulting multitude is again a set.*
(VI) Axiom of foundation: *In every non-empty set a, there is an element b that has no*

[14] See the Epilog for more details concerning implicit definitions.

element in common with a.

It is said that the concept of a *'set'* is *implicitly* defined by the system of these axioms, because it is 'woven' (or 'folded') into the system and, in this respect, has been limited in its meaning (Lat. *plicare* = to fold). - For more on *'implicit definability'* see the Epilogue.

One can prove, on the basis of these six axioms, that the familiar objects $\mathbb{N}$, $\mathbb{R}$, $\mathsf{P}(\mathbb{R})$, $\mathsf{P}(\mathsf{P}(\mathbb{R}))$, ... can be constructed as sets. However, a boundless formation of sets does not seem to be possible in the system ZSF, because there is no general axiom of comprehension. Thus, extensions of concepts cannot always be constructed as sets. Therefore, the absolute-infinite multitudes *V, Ω* and *Z* (see above) are not (at least, not in a direct way) constructible as sets on the basis of these six axioms.

One can (according to QUINE 1963) set up the system ZSF in such a way that one can talk about the absolute-infinite multiplicities *V, Ω* and *Z*, etc., but they are treated only *nominalistically*: they are *voces* and not *res*. The *finite and transfinite* multiplicities, however, are things (*res*). Thus, CANTOR's vision that only the *finite and transfinite* multitudes are *'sets'* could not be realized with an *explicit* definition, but it was possible to realize it with an *implicit* definition.

Set theory could, from then on, develop on the basis of the ZERMELO-SKOLEM-FRAENKEL axiom system *'zu einem der fruchtreichsten und kräftigsten Wissensgebiete der Mathematik"* (*'into one of the most fruitful and powerful fields of knowledge in mathematics'*) - cf. DAVID HILBERT, 1917. In particular, set theory could take on the role of providing a semantics for the structuralistically-conceived mathematics of the present time. We will come back to this in Chapter 19 and in our final considerations.

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Chapter 16
Contemporary Platonism

The term *'Platonism'* in the present Philosophy of Mathematics does not refer to the entire philosophy represented by PLATO, but only to a general tendency of Platonic philosophy. It was PAUL BERNAYS (1888-1977) who, in 1934 (op. cit.), gave this general tendency the name *'Platonism'*. It is *'the tendency to consider the objects of mathematics as cut from all links with the thinking subject'*, as BERNAYS has put it. However, we will have to distinguish carefully between the original *Platonic* doctrine (cf. Chapter 2) and the contemporary *Platonistic* standpoint with respect to the ontology and epistemology of mathematics.

Platonism, in the sense of BERNAYS, holds that the objects of mathematics exist *per se* independently of our own thinking and cognition. Mathematics describes a realm of immutable, predetermined objects that cannot be perceived by our senses, but, however,

> *"... which exists independently both of the acts and (of) the dispositions of the human mind and is only perceived, and probably perceived very incompletely, by the human mind."*
>
> KURT GÖDEL: *'Some basic theorems on the foundations of mathematics and their implications'* (Gibbs lecture), Collected Works III, p. 323.

In such a world of mathematical objects, all problems are already decided and, hence, every mathematical assertion is therefore either true or false. The mathematical propositions claim to express *facts*, i.e., facts that are true in the world of mathematical objects. PLATO indicated such a view in his dialogue '*Euthydemos*', 290c, as follows:

> *'For even the geometricians, the astronomers and the arithmeticians are hunters, because they do not make their figures and numerical series, but these are already, and they only find them as they are.'*

This view experienced a resurgence in the 19th and 20th centuries, especially in response to 19th century psychologism. PAUL BERNAYS, BERNARD BOLZANO, GOTTLOB FREGE, KURT GÖDEL, G. HAROLD HARDY, CHARLES HERMITE, WACLAW SIERPINSKI, ROBERT SOLOVAY, ERNST ZERMELO and many others explicitly declared their support for the Platonistic standpoint.

If one wants to adhere to the Platonistic programme, then one must do so only in set theory (including the axiom of choice), since, according to GÖDEL's completeness theorem of (1st order) predicate-logic, every consistent mathematical theory *T* has a model in the universe of all sets. The Platonistic standpoint is then also decisive for *T*. More generally, we may say, with GÖDEL, that *"all mathematics is reducible to abstract set theory"* (cf.

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U. Felgner, *Philosophy of Mathematics in Antiquity and in Modern Times*,
Science Networks. Historical Studies 62, https://doi.org/10.1007/978-3-031-27304-9_16

GÖDEL's Works III, p. 305). Therefore, in the following, we will only have to deal with the effects of the Platonistic point of view in set theory.

16.1 BOLZANO's Platonism

BOLZANO developed set theory from a Platonistic point of view. He believed that sets do not come into existence by forming them in our thoughts, but that we can only consider them because they are already present as things, independent of our thinking. BOLZANO wanted to say that sets are not entities that belong to *subjective* thinking, but that they have *objective* existence. (We have reported on this in Chapter 15.) He wrote,

> *"... und ich bemerke, dass die erwähnten Dinge mit Wahrheit nicht zusammengedacht werden könnten, wenn sie nicht auch, ohne dass wir sie noch zusammendenken, schon zusammen wären, schon ... ein Ganzes bildeten, ein Ding. ... Ich behaupte also, dass Inbegriffe bestehen nicht dadurch, dass wir sie denken, sondern umgekehrt, dass wir nur dann sie mit Wahrheit denken können, wenn sie bestehen auch ohne dass wir sie denken."*
>
> [... and I note down that the things mentioned could not be truely thought together if they were not, even without our thinking them together, already together, already... forming a whole, a thing. ... I therefore maintain that embodiments (sets) do not exist because we think them, but vice versa, that we can only truely think them if they exist even without us thinking them.]
>
> BERNARD BOLZANO: *'Größenlehre'*, op. cit., p. 100-101.

BOLZANO expressed a similar thought in his book on the *'Paradoxien des Unendlichen'*, Leipzig 1851, § 14, pages 15-17. He wrote very little to justify his opinion. He seemed to presuppose that the world does not only consist of unstructured, isolated material things, but that it has a structure, and therefore also consists of sets of material things and sets of sets of such things, etc. BOLZANO also seemed to presuppose (although he didn't talk about it) that we humans can perceive, with our senses and with our mind, all of these sets and sets of sets, etc.

In the *'Paradoxien des Unendlichen'* (1851, § 14, pp. 16-17), BOLZANO wrote that there is, for example, matter at the poles of the earth, and that these particles of matter are joined together to form lumps, chunks, ice floes, etc., and that these bodies obey the laws of physics, *'even if there is no human being, nor any other thinking being'* who observes all this. BOLZANO continued [1]:

> *"Bejaht man dieses - und wer müßte es nicht bejahen - dann ... gibt es also Mengen und Ganze, auch ohne dass ein Wesen, welches sie denkt, da ist."*
>
> [If one affirms this - and who would not have to affirm it - then ... there are sets and totalities, even without a being that thinks it, is there.]

Similar Platonistic positions were held by many other mathematicians in the 19th and 20th centuries. In his treatise *'Mathematical Proof'* (op. cit., p. 4) GODFREY HAROLD HARDY (1877-1947) wrote:

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[1] This reminds us of a dictum written down by AUGUSTINUS (cf. Chapter 8, at the end of section 8.3): *'That three times three is nine, ... is necessarily true, even then when all mankind snores.'*

"In some sense, mathematical truth is part of objective reality."

In his book *'A Mathematician's Apology'* (Cambridge Univ. Press 1967, p. 123), he wrote the same thing again, but with a few more details:

"I believe that mathematical reality lies outside us, that our function is to discover or observe it, and that the theorems which we prove, and which we describe grandiloquently as our ‚creations', are simply our notes for our observations. This view has been held, in one form or another, by many philosophers of high reputation from Plato onwards, and I shall use the language which is natural to a man who holds it." G.H. HARDY, 1967.

16.2 The usefulness of Platonism

BERNAYS considered *Platonism* to be the dominant view in mathematics at his time:

"il n'est pas exagéré de dire que le platonisme règne aujourd'hui dans la mathématique."
[It is no exaggeration to say that Platonism now *reigns* in mathematics.]
PAUL BERNAYS, op. cit.

This view still seems to be prevalent today. Its usefulness can be demonstrated with many examples from almost all areas of mathematics.

(1) Examples from Mathematical Logic.

(a) The hypothesis underlying the classical two-valued 'Propositional- and Predicate Logic', namely, that every statement from mathematics is either true or false, can be postulated as ARISTOTLE did (cf. Chapter 3), but can also be confirmed as valid on the basis of *Platonism*. This is because all mathematical statements describe facts that are either valid or not in the ideal, supernatural world of mathematical objects.

From the *thesis that all mathematical statements are either true or false*, the laws of the propositional calculus and predicate calculus are derivable as usual.

(b) In particular, the thesis gives rise to the principle of *tertium non datur* (principle of the excluded middle). It states that, for any statement Φ, the disjunction $\Phi \vee \neg\Phi$ (read: Φ or non-Φ) is always true. Also, the validity of the OCKHAM-DE MORGAN rules [2]

$$(\neg\Phi \vee \neg\Psi) \leftrightarrow \neg(\Phi \wedge \Psi) \text{ und } (\neg\Phi \wedge \neg\Psi) \leftrightarrow \neg(\Phi \vee \Psi)$$

can be confirmed with it.

(c) Additionally, the *existential quantifier* ‘There is (at least) one object *x* such that ... ’

[2] A historical remark: The *‘tertium non datur’* (principium exclusi tertii seu medii inter duo contradictoria - principle of the excluded third or middle between two contradictory statements) was first pronounced by ARISTOTLE in his *'Metaphysics'* (IV, 1011b24) and his *'Second Analytics'* (I, 71a14). The other two rules were first formulated by WILHELM OF OCKHAM (1285-1347) in his *'Summa Logicae, pars secunda'*, written in 1324/1325. They also appear later in the work *'De puritate artis logicae, tractatus brevior'* by WALTER BURLÄUS (ca. 1275-1345), written in 1326/1327. In the following centuries, they were rediscovered by GOTTFRIED WILHELM LEIBNIZ (1646-1716) and others, and finally also by AUGUSTUS DE MORGAN (1806-1871).

(in symbols: $\exists x$ ...) is equivalent to the expression 'It is not the case that, for all objects x, it is false that ... '. (in symbols: $\neg \forall x \neg$ ...). So, it suffices to prove the existence of objects satisfying certain properties *indirectly* in order to be sure that such objects really exist. According to *Platonism*, all objects in question already exist and the problem of whether they can be brought into existence does not arise. (We will discuss the problem of non-constructive proofs of existence in more detail in Chapter 17).

(d) *Impredicative definitions* are allowed. HENRI POINCARÉ had labeled the definition of an object g to be *'impredicative'* if it contains a quantifier that runs through all objects of a domain B to which g also belongs. POINCARÉ argued that, in such a definition, the quantifier would also run over the object g, and that, in this respect, g is not explicitly defined. The definition of g uses the existence of g and is, in this respect, according to POINCARÉ, circular - see POINCARÉ: *'Sechs Vorträge'*, op. cit., p. 47. According to the opinion of the Platonists, however, it is permissible to define an object by specifying a property that makes it stand out (or distinguishes it) among all *existing* (!) objects.

(e) The following example of an impredicative definition was given by ZERMELO. (It can be found in POINCARÉ's 5th lecture, loc. cit., p. 47.) Let $f(x)$ be any non-constant polynomial with real or complex coefficients. One proves that $|f(x)|$ must have a minimum, and let c be a complex number such that $|f(c)|$ is minimal. In 1813, JEAN-ROBERT ARGAND (1768-1822) was able to show, in a direct way, that, for such a number c, even $f(c) = 0$ holds. With this, he had succeeded in providing a new perfect and simple proof of the famous *fundamental theorem of algebra*. It was the very first complete and correct proof of that fundamental theorem! But, in this proof, c is defined impredicatively (cf. also Chapter 17 for some further remarks).

(2) An example from algebra.

(f) In his book *'Transzendente Zahlen'* (Mannheim 1967, p. 72), CARL LUDWIG SIEGEL noticed the following fact:

Theorem: *At least one of the two numbers* $e + \pi$ *or* $e \cdot \pi$ *is transcendent.*

Proof. We consider the polynomial $(x - e) \cdot (x - \pi) = x^2 - (e + \pi)x + e\pi$. If both $e + \pi$ and $e\pi$ were algebraic numbers, then $x^2 - (e + \pi)x + e\pi$ would be a polynomial with coefficients in the field A of all algebraic numbers. However, A is algebraically closed and, consequently, the two roots e and π would also have to be in A. But, according to CHARLES HERMITE (1873), e is transcendent and, according to FERDINAND LINDEMANN (1882), π is also transcendent - a contradiction, Q.E.D.

Strictly speaking, the proof only provides the information that it is wrong to claim that both, $e + \pi$ and $e\pi$, are algebraic numbers. The sentence "$e + \pi$ is algebraic *and* $e\pi$ is algebraic" is therefore wrong, and thus has a truth value. But it does not follow from this that the two individual sentences "$e + \pi$ is algebraic" and "$e\pi$ is algebraic" also have truth values. If one takes the Platonist point of view, then both sentences have truth values and one can use the OCKHAM-DEMORGAN rule $\neg(\Phi \wedge \Psi) \leftrightarrow (\neg\Phi \vee \neg\Psi)$ and conclude that one of the two negated sentences must be true (compare this with (b)).

(3) Examples from set theory.

(g) The existence of actual infinite sets can be doubted, and has been doubted occasionally (cf. Chapters 5 and 6). But, since there are obviously finite sets of any size, these sets altogether form (from a Platonist point of view) an infinite set.

(h) It can be questioned whether, for every infinite set M, the so-called power set $\mathsf{P}(M)$ always exists as a 'completed gathering' of all subsets of M. GEORG CANTOR considered its existence to be evident. RENÉ BAIRE (1874-1932) denied its existence in a letter to JACQUES HADAMARD (1865-1963) (Letter II in the *'Cinq lettres'*, op. cit.). If one takes the point of view of *'Platonism'*, then the existence of $\mathsf{P}(M)$ is secured for every set M.

(i) From the point of view of *'Platonism'*, the *Axiom of Pairing,* the *Axiom of Union* and the *Axiom Schema of Replacement* of ZERMELO-SKOLEM-FRAENKEL's Set Theory ZSF (also denoted as ZF) are also true.

In order to justify the *axiom of choice,* AC, one relies additionally on the viewpoint that an *omnipotent and omniscient Being* must be able to select, from each set of a set of pairwise disjoint non-empty sets, one element and to combine the selected elements into a further set.

However, the *Generalized Continuums-Hypothesis* GCH ($\forall\alpha$: $2^{\aleph_\alpha} = \aleph_{\alpha+1}$) cannot be justified from the point of view of *Platonism*. So, *Platonism* has its limits.

16.3 The restricted (or weak) Platonism

One must not interpret Platonism as unrestricted conceptual realism, because, otherwise, the antinomy of ZERMELO-RUSSELL (cf. Chapter 15) would be an undesired consequence. The limit is to be drawn in such a way that, for all finite and all transfinite sets (in the sense of CANTOR), the standpoint of the Platonists can be taken up, but not for the absolute-infinite multitudes. PAUL BERNAYS formulated the following weakened version in 1935:

The *thesis of a restricted (*or *weak) Platonism*: All individual integers, the set $\mathbb{Z}$ of all integers, and their power sets $\mathsf{P}(\mathbb{Z})$, $\mathsf{P}(\mathsf{P}(\mathbb{Z}))$, $\mathsf{P}(\mathsf{P}(\mathsf{P}(\mathbb{Z}))$, ... (finitely often iterated) do exist independently of the thinking subject.

For the needs of the so-called *'Analysis'* (i.e., the theory of real and complex functions and the differential and integral calculus), this restricted form of Platonism is sufficient.

16.4 GÖDEL*'s* Platonism

GÖDEL was probably the only one who ever tried to defend his Platonist standpoint with strong arguments. Let us first briefly introduce GÖDEL.

KURT FRIEDRICH GÖDEL was born on April 28, 1906, in the Bohemian-Austrian city of Brünn, called Brno since 1919. He spent his school years there from 1912 to 1924, and, in 1924, he moved to the University of Vienna to study mathematics and physics. He received his doctorate in 1929, at the age of 23, with the proof of the completeness of the HILBERT-

ACKERMANN predicate calculus, thus solving a problem stated on page 68 of the *'Grundzüge der Theoretischen Logik'* (Springer Verlag Berlin, 1928) by D. HILBERT and W. ACKERMANN.

GÖDEL habilitated at the University of Vienna in 1932, with a paper in which he showed that the axiom system of first-order arithmetic, DPA, given by DEDEKIND and PEANO, and also any recursive extension DPA* of DPA (in the same language), is incomplete, provided that DPA is ω-consistent. (JOHN BARKLEY ROSSER proved, in 1936, that, instead of ω-consistency, one need only assume the (usual) consistency of DPA). Thus, for each such extension DPA*, there are always number-theoretical statements that are neither provable nor refutable in DPA*. This theorem is called *'*GÖDEL*'s first incompleteness theorem'*. GÖDEL also conjectured that even the statement *Wid*(DPA), which expresses the consistency ('Widerspruchsfreiheit') of the DPA system (in number theoretically coded form), is not provable in the DPA system, provided DPA is consistent. This is the *'second incompleteness theorem'*. However, GÖDEL never arrived at a complete proof of that theorem. The first complete proof was given in 1939, by PAUL BERNAYS, and published in vol. 2, pp. 293ff, of the HILBERT-BERNAYS *'Grundlagen der Mathematik'*.

In the years 1935-1939, GÖDEL succeeded in proving the following highly important theorem: If the usual *Zermelo-Skolem-Fraenkel* set theory ZSF is free of contradictions, then the set theory ZSF extended by the axiom of choice AC and the Generalized Continuums-Hypothesis GCH is also free of contradictions.

In January 1940, K. GÖDEL accepted an invitation to the Institute for Advanced Study in Princeton, NJ, USA. From that point on, his scientific work shifted almost exclusively to the field of philosophy. In a series of treatises, he developed his conviction *that mathematics has a content that precedes any formalization and that this content cannot even be completely formalized.*

As early as the '30s, GÖDEL was already occasionally suffering from exhaustion and nervous breakdowns, which made extended stays in sanatoriums necessary. At the end of his life, his depressions intensified. For fear of being poisoned, he ultimately refused to eat. GÖDEL died at Princeton Hospital on January 14, 1978.

KURT GÖDEL strongly supported Platonist realism; he considered Platonism the only justifiable position. What he understood by Platonism, he described (in his GIBBS *Lecture* 1951, cf. GÖDEL*'s Collected Works*, op. cit., Volume 3, pp. 322-323) with the following words:

> *„I am under the impression that ... the Platonistic view is the only one tenable. Thereby I mean the view that mathematics describes a non-sensual reality, which exists independently both of the acts and (of) the dispositions of the human mind and is only perceived, and probably perceived very incompletely, by the human mind. This view is rather unpopular among mathematicians; there exists, however, some great mathematicians who have adhered to it. For example, Hermite once wrote the following sentence: Il existe, si je ne me trompe, tout un monde qui est l'ensemble des vérités mathématiques, dans lequel nous n'avons accès que par l'intelligence, comme existe le monde des réalités physiques; l'un et l'autre indépendants de nous, tous deux de création divine".*

[HERMITE: 'There exists, unless I am mistaken, an entire world consisting of the totality of mathematical truths, which is accessible to us only through our intelligence, just as there exists the world of physical realities; each one is independent of us, both of them divinely created.']

The translation of the quotation from HERMITE is by SOLOMON FEFERMAN, with the assistance of MARGUERITE FRANK. - cf. GÖDEL's *'Collected Works'*, vol. III, p. 304 & p. 323. The quotation itself is contained (on p. 142) in the book *'Charles Hermite, Éloges académique et discours',* edited by JEAN-GASTON DARBOUX, Paris 1912.

GÖDEL wrote very cautiously that the human mind is able to perceive mathematical objects, but *"this is probably also only very incomplete"*. That doesn't sound as if GÖDEL could have said anything precise about the extent of these *'very incomplete mental perceptions'*. GÖDEL*'s* hints are very reserved here.

It is not unproblematic to assert that there is a world in which mathematical truths are realized, and, even if such a world were to exist, that we might have the ability to comprehend anything about it beyond all ratio. Even if this world were *objectively* present, the ability to recognize it would, at best, only be *subjectively* given, and it would be very doubtful whether we could *teach and learn* these abilities. The mathemata (μαθήματα) are, according to the original meaning of the word, as we have seen in Chapter 1, the objects of teaching and learning, not through sensory or extrasensory experience, but through understanding. It is still true today that mathematics must not refer to sensory or extrasensory experience, but only to understandable, comprehensible and always repeatable proofs.

If there were extramental mathematical entities, then there would have to be an appropriate epistemology by which to view and study the properties of these entities and the relations between them. But there is no such epistemology. It is more than doubtful whether our mind (or reason) has the ability to 'see' mathematical truths beyond all ratio. We (presumably?) have no higher senses to perceive abstract entities that do not belong to the spatio-temporal world.

Nevertheless, many mathematicians have declared their support for *Platonism*, although KURT GÖDEL was probably the only one who tried to prove that mathematics as a whole could not be our creation, and that it is therefore reasonable to assume that the world of mathematical objects is given to us and

> *"that our function is to discover and observe it, and that the theorems which we prove, and which we describe grandiloquently as our 'creations', are simply our notes for our observations",*

as HARDY had put it (see above).

16.5 GÖDEL*'s* vindication of Platonism

In GÖDEL*'s* later philosophical treatises, one repeatedly comes across statements that support his Platonist views. For example, in 1944, he wrote, in his essay on *'Russell's Mathematical Logic'* (GÖDEL's *'Collected Works',* III, p. 137):

> *"It seems to me that the assumption of such objects [i.e. classes and concepts] is quite as*

legitimate as the assumption of physical bodies and there is quite as much reason to believe in their existence.“

In his 'GIBBS *Lecture*', 1951, GÖDEL tried to prove that the objects of mathematics cannot be our own creations (cf. *'Collected Works'*, vol. III, pp. 290-323). GÖDEL was convinced that mathematics is about contents that are not created by us humans and that, for us, are predetermined. However, it is - as we think - questionable whether such contents exist independently of our own thinking and imagination and how we could perceive them. GÖDEL meant that we can at least partially perceive them with our mind. In the handwritten notes for his 'GIBBS lecture', GÖDEL compared the perception by our mind (or 'reason') with the perception by a sixth sense (cf. E. KÖHLER, op. cit., p. 343).

We do not know whether we can perceive anything at all with the help of our mind that does not come from the senses or from our own imagination, and we leave this question undiscussed. But we want to pursue the question as to whether the contents of the various mathematical theories are our own creations, or whether these contents are predetermined for us.

In his 'GIBBS-lecture', GÖDEL starts from the thesis that *the human brain* can be equated with a *machine* (a finite machine, or, more precisely, a TURING-machine). Through appropriate coding (also called *‘Gödelisation’*), all production processes of such a machine can be understood as a subsystem of a recursive extension of the DEDEKIND-PEANO-arithmetic DPA. However, such a system of arithmetic allows for (as the proof of the first of GÖDEL's incompleteness theorems shows) the formulation of unprovable and irrefutable number-theoretical statements. Such statements can be put into the form of Diophantine equations.

Consequently, if the *human brain* can be equated with a *machine*, then there would have to be Diophantine equations that are unsolvable for it. We can record the result of this discussion as follows:

(*) *If* ‘the human brain’ *can be equated with a machine, then there are Diophantine equations that the brain cannot solve.*

GÖDEL then asks the question as to whether the *human mind* can be reduced to the *human brain.* If this is the case, then the mind cannot contribute anything to the solution of Diophantine equations that cannot already be done by the brain. There would then even be *absolutely unsolvable* Diophantine equations. 'HILBERT*'s optimism*' that there is no *ignorabimus* in mathematics (cf. Chapter 20) would not be justified. So:

(**) *If* the human mind *is reducible to the brain, and can therefore be equated with a machine, then there are absolutely insolvable Diophantine equations.*

GÖDEL argues that the conclusion of this implication (**) also has highly interesting consequences. He means,

(†) *that it follows from the existence of absolutely unsolvable Diophantine equations that mathematics as a whole cannot be our creation.*

GÖDEL's proof of (†)[3]: If it were our own creation, then we, as creators of all mathematical objects (especially as creators of all natural numbers and their arithmetic), would have to know everything about these objects. In particular, we would, in principle, have to be able to solve every Diophantine equation, and hence there would be no absolutely unsolvable Diophantine equations - a contradiction.

From the results (**) and (†), the following claim is now immediately apparent:

(#) *Either the human mind is not reducible to the brain, or mathematics is not entirely created by the human mind (or both alternatives are true at the same time).*

The first alternative of (#) expresses the non-sensuality of the mind and allows for the *possibility of metarational cognitions*, as Platonism assumes.

According to GÖDEL, the second alternative of (#) expresses the inadequacy of all views that are not entirely free of subjectivity, confirming the objectivity of mathematics, and thus, as he believes, Platonism.

So, both alternatives give hints, as GÖDEL thought, that the position of Platonism cannot be rejected outright. GÖDEL even considers Platonism to be the only reasonable position and considers his argumentation to be a valid vindication of the thesis that mathematics cannot be entirely our creation.

16.6 Discussion

GÖDEL's argumentation is quite impressive, but not everyone will find it convincing.

However, GÖDEL anticipated some of the possible objections. In reference to his proof of (†), he wrote (*'Collected Works'* III, p. 312) that someone who constructs, for example, a car, an airplane or a bomb will not necessarily know all characteristics of the object that they have created. GÖDEL meant that the creator of such an object uses materials and tools that are not his own creations. In contrast to this, someone in mathematics who, e.g., considers himself the creator of natural (or irrational) numbers would have to create these numbers using only tools of his own rational mind, and without any additional material.

But other objections can be made.

GÖDEL uses the terms *'brain', 'mind', 'machine'* and *'creator'*, etc., rather uncritically, so that the validity of his arguments suffers altogether (ECKEHART KÖHLER has already pointed this out, op. cit.). Exactly proven theorems of mathematical logic (GÖDEL's incompleteness theorems) and exactly defined mathematical terms ('recursivity', 'unprovability in the DPA system', etc.) are associated with colloquial terms that have no sharp contours. But no clear insights can be gained from such a connection.

GÖDEL's argument is therefore not compelling. It is not even tempting, because, in mathematics, one does not want to have to refer to the possibility of uncontrollable meta-rational insights.

The views held by *Platonism* may have their appeal, but they are not convincing.

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[3] The proof follows an old argument by ZENO of Kition, which is also to be found in SEXTUS EMPIRICUS (*'Adv. Math.'* VII, 248), CICERO (*'Academica post.'* I,§XI,Nr.42), ALEXANDER PICCOLOMINI (cf. Ch. 7) and RENÉ DESCARTES (cf. Ch. 8). We discussed it in detail in Chapter 8, section 8.6.

Whether there is a world of mathematical objects that exists independently of our thinking and cognition remains pure speculation. And, if this world existed, could we - beyond all ratio - perceive anything of this world? The answers here are probably all more or less negative.

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Chapter 17
The Problem of non-constructive Proofs of Existence

"Come l'araba fenice, Che vi sia ciascun lo dice
...Dove sia... nessun lo sa."

[... it's like the Arabian phoenix; that it exists, everyone says, but where it is, no one knows.]

LORENZO DA PONTE /WOLFGANG AMADEUS MOZART*:*
'Così fan tutte', Act 1, Scene 1, No. 2: Terzetto.

We want to discuss the question as to what we can expect from a proof of existence in mathematics. Are we allowed to expect that an object that fulfills the claim of existence is explicitly given by name and also clearly described, - or do we have to rest satisfied with the hint that the assumption of non-existence of such an object would lead to contradictions? So, the question is which information the existential quantifier $\exists x$ (read: *there exists an x*) can and should provide.

This question was discussed very fiercely and controversially at the beginning of the 20th century in connection with the heated 'foundational debate' and the emergence of Platonism and Intuitionism. We want to follow this discussion. - An example should first make clear what this discussion is about.

17.1 The *'existence'* of transcendental real numbers

Theorem (JOSEPH LIOUVILLE, 1844): *There are transcendental numbers, i.e., real numbers, that are not algebraic.*

LIOUVILLE (1809-1882) proved this theorem of existence at a time when not a single example of a transcendental real number was known. That EULER*'s* number $e = 2{,}7182\ ...$ is transcendental was not proved until 1873, by CHARLES HERMITE (1822-1901). That the circle number $\pi = 3{,}141\ 592\ 653\ ...$ is also transcendental was shown by FERDINAND LINDEMANN (1852-1939) in 1882.

First Proof (LIOUVILLE): It is shown in a first step that irrational numbers r, for which there is an infinite sequence of rational numbers a_n/b_n ($1 \leq n \in \mathbb{N}$; a_n , $b_n \in \mathbb{N}$; a_n and b_n relatively prime) with the property

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(#) $$\forall n \left(1< b_n \;\; \& \;\; |r - a_n/b_n| < (1/b_n)^n\right),$$

are transcendental. In a second step, one concludes from this that, for example, the following number (whose decimal development is apparently not periodic, and is therefore irrational) is transcendental (here, n! denotes the n^{th} factorial, e.g., 4!=24):

$$10^{-1!} + 10^{-2!} + 10^{-3!} + ... = 0{,}11000\ 10000\ 00000\ 00000\ 00010\ 00...\ ,\ \text{Q.E.D.}$$

The proof that real numbers, which satisfy LIOUVILLE's criterion (#), are transcendental is somewhat complex. LIOUVILLE relies on the fundamental theorem of algebra (see below, section 17.2) and a theorem about a certain approximation of algebraic numbers by rational numbers (see TH. SCHNEIDER: *'Einführung in die transzendenten Zahlen'*, Springer Verlag Berlin 1957, pp. 1-3). As soon as all of this is proven, one can explicitly specify infinitely many transcendental numbers and call them by name in the decimal place value system.

This proof was admired by all mathematicians who made an effort to study it in detail. That the theorem also has an easy and rather short proof was quite surprising:

Second Proof (GEORG CANTOR - RICHARD DEDEKIND, 1873/1874): The set of all algebraic numbers is countable (according to DEDEKIND[1]) and the set of all real numbers is uncountable (according to CANTOR, cf. Chapter 15). So, not every real number is algebraic, and consequently there must exist real numbers that are transcendental. Q.E.D.

In contrast to LIOUVILLE*'s* proof, the CANTOR-DEDEKIND proof does not depend on intricate number-theoretic arguments. It only depends on a simple, but ingenious, *'diagonal argument'* (in the sense of PAUL DU BOIS-REYMOND, cf. Chapter 15, footnote 5).[2] It even follows that there must be more transcendental numbers than algebraic numbers. *But, in contrast to LIOUVILLE's proof, the CANTOR-DEDEKIND proof does not give any hint as to how to explicitly state even a single transcendental number!* The CANTOR-DEDEKIND proof only shows that the set of all real algebraic numbers does not exhaust the totality of all real numbers and that - in principle - there must be real numbers that are not algebraic, and hence transcendental.

The confrontation of both proofs shows that, in mathematics, proofs of existence can have very different meanings.

- They can mean that it is only shown that the objects in question must exist in the sense, that it would be contradictory to assume their non-existence. We are inclined to call such proofs of existence *non-constructive proofs of existence,* because they do not give any hint as to how to explicitly name or describe certain objects whose existence is claimed. The CANTOR-DEDEKIND proof is an example of such a non-constructive proof of existence.

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[1] DEDEKIND had proved this on December 1, 1873, and informed CANTOR. CANTOR then succeeded in proving, on December 7, 1873, that the set of all real numbers is uncountable, and published both results in CRELLE's Journal (vol. 77 (1874), pp. 258-262) under his own name. He did not mention DEDEKIND*'s* name anywhere. DEDEKIND was disappointed with CANTOR*'s* behavior, but did not protest publicly.

[2] Notice that the approximation by rational numbers also makes use of a special kind of diagonalization.

- But they can also mean that the existence of the objects in question is shown by explicitly naming some of them, or by describing some of them in sufficient detail so that everyone knows how to construct (or reconstruct) them in finitely (or infinitely) many steps. We are inclined to call such proofs *effective* or *constructive existence proofs.*

But what about the proof given by LIOUVILLE? Is that proof constructive? It seems so, since LIOUVILLE was able to explicitly name some irrational numbers that are transcendental. However, in his proof, he used the fundamental theorem of Algebra, and so the question is whether that theorem has a constructive proof. The first proofs, given by J.L. LAGRANGE 1772, P.S. LAPLACE 1795/1812, C.F. GAUSS 1799, 1815, 1816, 1849, R. ARGAND 1814, A.L. CAUCHY 1821 and others, had been non-constructive. But a few decades later, the non-constructive steps in the proofs by ARGAND and CAUCHY could be replaced with constructive steps. Such modifications have been found by CARL WEIERSTRASS (1859, 1891), HELMUT KNESER (1940), MARTIN KNESER (1981) and M.W. HIRSCH & STEVE SMALE (1979, op. cit.). We will discuss the history of the efforts towards obtaining constructive proofs of that theorem in the following section.

17.2 The *'existence'* of roots of polynomials

How to find, construct and describe the roots of polynomials of the second, third and fourth degrees (with real coefficients) was shown by the Greek, Persian, Italian, German and French mathematicians: EUCLID (ca. 300 B.C.E.), AL-KHWOÁRIZMI (ca. 825), OMAR KHAYYAM (ca. 1070), SCIPIONE DAL FERRO (1515), NICOLO TARTAGLIA (1535), GERONIMO CARDANO (1545), LUDOVICO FERRARI (1545), RAFAEL BOMBELLI (1572/1579), MICHAEL STIFEL (1544), FRANÇOIS VIÈTE (1615), JOHANNES FAULHABER[3] (1622), RENÉ DESCARTES (1637) and others. They gave algebraic, analytic and sometimes geometric descriptions of the roots. For a long time, the only controversy was whether the complex roots are to be considered as 'true' roots, and also whether there are further roots outside of the field of complex numbers.

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[3] In his small booklet *'Miracula arithmetica. Zu der Continuation seines Arithmetischen Wegweisers gehörig'*, 99 pages, published 1622, in Augsburg, FAULHABER (1580-1635) gave (on p. 67) an elegant new method of describing the roots of biquadratic polynomials, i.e., polynomials of the 4th degree (in FAULHABER's terminology, *Aequationen der Zenßzenß Coss*). When a biquadratic polynomial $f(x) = x^4 + px^2 + qx + r = 0$ is given, then the factorization $f(x) = (x^2 + yx + b)(x^2 - yx + c)$ allows us to express p, q and r in terms of y, b and c, which gives rise to a cubic resolvent $g(z) = z^3 + 2pz^2 + (p^2 - 4r)z - q^2 = 0$ (with $z = y^2$), which can be solved according to CARDANO. If θ is a root of $g(z)$, then $y = \sqrt{\theta}$, and the terms b and c can also be calculated. From these terms, the roots of the two quadratic polynomials, and hence also the four roots of $f(x)$, can be explicitly written down.
Notice that EULER simplified the representation of the four roots in his *'Vollständige Anleitung zur Algebra'* from 1770 (Part 2, Chapter 15 - cf. EULER's *Opera Omnia*, Series I, vol. 1, p. 309).
Notice also that DESCARTES, in his booklet *'La Géométrie'* from 1637, gave a brief discussion of the method of solving biquadratic polynomials. He mentions FAULHABER's cubic resolvent $g(z)$, however, without any explanation, and also without mentioning FAULHABER's name (cf. DESCARTES, *Œuvres VI*, pp. 457-458).

For polynomials of the fifth degree or higher (with real coefficients), no one was able to prove the existence of roots, despite intensive efforts. No one could find either geometric constructions or algebraic descriptions of the roots of these polynomials. In fact, the results of PAOLO RUFFINI (1799), NIELS HENRIK ABEL (1826) and ÉVARISTE GALOIS (1832) show that the general polynomial equation of degree $n \geq 5$ is not solvable by radicals. This means, that its roots, *if there are any,* cannot be calculated by a *finite* sequence of rational operations and the extraction of roots, beginning with the coefficients of the polynomial.

The question arises as to whether the roots, *if there are any,* can be found as the limits of *infinite* convergent sequences of complex numbers, and that, hopefully, in a constructive way. This question actually consists of two questions: firstly, *'whether there are roots',* and secondly, whether these roots can be calculated, i.e., can be found through constructive procedures.

The first question was answered in the decades around 1800, when quite a number of different proofs appeared for the theorem that, in fact, *every non-constant polynomial with real (or complex) coefficients has at least one root and that all its roots can be found in the field of complex numbers*. This is the famous *fundamental theorem of Algebra.* Its authors included D'ALEMBERT (1746/1748), EULER (1749/1751), LAGRANGE (1772), LAPLACE (1795/1812), GAUSS (1799, 1815, 1816, 1849), ARGAND (1814), CAUCHY (1821) and others. However, most of these proofs contained more or less serious gaps, but, in the years 1814/1815, the first complete and correct proofs appeared in print (by ARGAND and GAUSS). All of these complete or incomplete proofs had been non-constructive. These proofs claim the *mere existence* of roots, and that they are located in the field $\mathbb{C}$ of the complex numbers. However, no one knew where to find them. - *It's like the Arabian phoenix - that it exists, everyone says, but where it is, no one knows.*

We mentioned already that, a few decades later, the non-constructive steps in the proofs by ARGAND and CAUCHY were replaced with constructive steps. Such modifications were found by CARL WEIERSTRASS (1859, 1891), HELMUT KNESER (1940), MARTIN KNESER (1981) and M.W. HIRSCH and STEVE SMALE (1979, op. cit.). Further details can be found in the essay *'Fundamentalsatz der Algebra'*, op. cit., by REINHOLD REMMERT (1930-2016).

So far, we have seen a few theorems in which the existence of certain mathematical objects is asserted and which originally had been proved in a non-constructive way, but were later able to be equipped with constructive proofs. Is this what we have to expect from all other non-constructive existence proofs, namely, that they are worthless if it is impossible to ameliorate them with constructive proofs? In order to find a convincing answer, we should look into the history of mathematics and examine the situation in earlier times.

17.3 Proofs of existence in ancient mathematics

From antiquity on up to the 19th century, proofs of existence were mostly constructive. The few non-constructive proofs of existence were controversial. ARISTOTLE called them

'eristic' (i.e., *quarrelsome*) in the *'Sophist Elenches'* (171b12-19),[4] and RENÉ BAIRE even called them *'trompeuse'* (i.e., *fraudulent*) in the 2nd letter of the *'Cinq Lettres',* op. cit., p. 264. There had also been a number of mathematicians who did not want to accept non-constructive proofs of existence. Therefore, let us start our investigation with an examination of what the mathematicians of antiquity demanded of a *proof of existence*. How did they deal with the epistemological problem of how to prove claims of existence?

HIERONYMUS ZEUTHEN (1839-1920) claimed, in 1896 (op. cit.), that, in ancient times, only geometric constructions had served as proofs of existence. OSKAR BECKER adopted this thesis in his book *'Mathematische Existenz'* (1927, op. cit., page 469) and wrote that

> *"... seit der Antike ... das Kennzeichnende des existierenden mathematischen Gegenstandes in seiner Konstruierbarkeit gelegen* (habe)".
>
> [... since antiquity... the distinguishing feature of the existing mathematical object had been its constructability.]

WILBUR KNORR, VOLKER PECKHAUS, ARTHUR DONALD STEELE and CHRISTIAN THIEL put forward convincing arguments against this claim (see the references below). In particular, they emphasized that there is far too little evidence to support such a claim. The few surviving texts rather indicate that mathematicians in ancient times were convinced that mathematical objects are not our creations. They exist independently of whether we observe them or not. In other words: they exist independently of whether we (re)construct them or not. In PLATO's dialogue *'Euthydemos'*, 290c, for example, it says:

> *'For even the metrologists, the calculators and the astronomers are hunters, because they do not make their figures and numerical series, but these are already, and they only find them as they are.'*

PROCLUS (ca. 411-485) wrote, in his Commentary on EUCLID's *'Elements'* (78F; Preface, Part 2), that, according to the opinion of the academic SPEUSIPP (he died ca. 340 B.C.E.), mathematicians do not create geometric configurations, because geometric objects all have an eternal existence; they all are uncreated and imperishable. A geometrician who constructs an equilateral triangle (see EUCLID *'Elements'* **1**,1) or a regular pentagon (EUCLID *'Elements'* **4**,11) does not create it, but only brings the individual parts one after the other in front of the mental eye, in order to *'look at them in a recognizing way'.* PROCLUS reports (again, in the 2nd part of the preface) that MENAICHMOS (he lived in the middle of the 4th century B.C.E.) also believed that geometric figures only seem to emerge during construction, but that *'in our mind everything exists (already) without becoming and without any change'.*

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[4] The word alludes to Eris, the Greek goddess of discord. According to the legend, because she was not invited to the wedding of Thetis and Peleus, she threw a golden apple with the inscription 'to the fairest' among the wedding guests. This led to a dispute between Aphrodite, Athena and Hera. Each claimed the apple for herself. They chose the beautiful shepherd Paris to decide the dispute. He chose Aphrodite, because she had promised him Helena, the fairest woman, for his wife. With Aphrodite's help, he seduced Helena and carried her off to Troy. Hence arose the celebrated Trojan War, which lasted 10 years.

JOHANNES PHILOPONOS (he lived around 550 C.E.) wrote, in his commentary on the Aristotelian *'Second Analytics'*, that mathematicians were not concerned with proving the existence of mathematical objects, but only with the practical execution of the configurations in question, assuming that they existed. He wrote, with regard to the problem of squaring the circle:

> *'Those who are working on the problem of squaring the circle don't investigate whether it is at all possible for a square to be equal to a circle, but rather, assuming that this is possible, are trying to actually construct a square which has the same area as the circle'.*
>
> PHILOPONOS: *'In Analytica posteriora',* ed. M.Wallies, CAG XIII3, Berlin 1909, 112: 25-36.

From these quotations, we can see that, in antiquity, the prevailing opinion was that all mathematical objects are predetermined for us and that they do not acquire their existence only at the moment of (re-)construction.

The problem of how to understand claims of existence of mathematical objects has only been controversially discussed in modern times. The discussion gradually began in the context of efforts to prove the so-called fundamental theorem of algebra. Here, constructive proofs had been received with much more satisfaction than non-constructive proofs. Surprisingly, in the course of the 19th century, the situation changed. This change was brought about by a new, brilliant short proof of a number-theoretical theorem given by CARL FRIEDRICH GAUSS. We will report on this in the next section.

17.4 GAUSS: *'notio'* or *'notatio'*?

In the *'Disquisiones Arithmeticae',* §77, by C.F. GAUSS (Leipzig, 1801), there is the remark that, in mathematics, it can often be advisable to *extract truths from concepts* (or notions) *rather than from formulas* (notations). There, GAUSS gives a proof of the so-called *theorem of Wilson* that, for every prime number $p \geq 2$, the number $1+(p-1)!$ is divisible by p, or, in other words, that

$$(p-1)! \equiv -1 \text{ (modulo } p).$$

This theorem was first noticed by LEIBNIZ (1646-1716) (see LEIBNIZ*'s 'Math. Schriften'*, Volume 7, pp. 180-181). EDWARD WARING erroneously ascribed it to Sir JOHN WILSON (1741-1793). The first proof was found by JOSEPH LOUIS LAGRANGE in 1768, and published in 1771 (*'Oeuvres de LAGRANGE'*, edited by SERRET, volume 3, Paris 1869, pp. 425-438).

First Proof (LAGRANGE, 1768) Let p be any prime number. The starting point here is the 'theorem of FERMAT':

$$\forall x\,(1 \leq x < p \;\Rightarrow\; x^{p-1} \equiv 1 \text{ (mod. } p)).$$

The numbers 1, 2, 3, 4, ... , p-1 are thus the pairwise incongruent roots of the polynomial $x^{p-1}-1$. With EUCLID's algorithm, it can be shown that $x^{p-1}-1$ splits into a product of linear polynomials $x-k$ (for $1 \leq k < p$). The two polynomials $x^{p-1}-1$ and $\prod_{k=1}^{p-1}(x-k)$ are thus congruent modulo p (i.e., they have the same degree, and the coefficients of the terms x^k (for $0 \leq k \leq p-1$) are each congruent modulo p). But the constant term of the polynomial

$\prod_{k=1}^{p-1}(x-k)$ is $(p-1)!$ and the constant term of $x^{p-1}-1$ is -1. Consequently, $(p-1)! \equiv -1$ (modulo p), as claimed.

Second Proof (GAUSS, 1801) The starting point here is the fact that the ring $\mathbb{Z}$ of integers, taken as modulo p, is a *'field'* (in the sense of Algebra),[5] and that each element a different from 0 has a uniquely determined multiplicative inverse a^*, i.e., $a \cdot a^* \equiv 1 \pmod{p}$. We have $1^*=1$ and $(p-1)^* = p-1$. But $a \neq a^*$ for $2 \leq a < p-1$. So, with $A = \{a;\ 2 \leq a < a^* \leq p-1\}$: $(p-1)! = 1 \cdot 2 \cdot \ldots \cdot (p-1) = 1 \cdot \prod_{a \in A}(a \cdot a^*) \cdot (p-1) \equiv -1$ (*modulo* p) , Q.E.D.

The Gaussian proof is short and elegant. In contrast to LAGRANGE's proof, in which all steps of the proof consist of calculations, the Gaussian proof avoids calculations. The Gaussian proof is *combinatorial,* because, in it, a is *'combined'* with its 'arithmetic inverse' a^*. This argument, by the way, is reminiscent of the argument with which the 8-year-old GAUSS amazed his teacher when he (the teacher) gave him the task of adding all numbers from 1 to 100 together. GAUSS immediately obtained the result

$$\sum_{k=1}^{100} k = 100 + (1+99) + (2+98) + \ldots + (49+51) + 50 = 50 \cdot 100 + 50 = 5050.$$

But, in the determination of the sum $\sum_{k=1}^{100} k = 5050$, one *only calculates,* while, in the determination of the product $(p-1)! = 1 \cdot (\prod (a \cdot a^*)) \cdot (p-1)$, one argues *conceptually* (i.e., using *'notions'*)! For the 'inverse' element a^*, there is no arithmetic expression that describes it, no *'notation',* as GAUSS put it; the 'inverse' element *exists* only according to the concept of a 'field'. - In the determination of the sum 1 + 2 + ... + 100, the law of pair formation can be specified explicitly. In the determination of the product $(p-1)!$, however, the pairings can only be given *according to the concept* (but not by a formula).

The Gaussian proof is therefore also *algebraic,* because, in it, the concepts of *'field'* and 'inverse element' (with respect to the field multiplication) are the essential concepts.

GAUSS commented on his beautiful proof as follows:

> *"At nostro quidem judicio hujusmodi veritates ex notionibus potius quam ex notationibus hauriri debebant."*
>
> [In our opinion, however, such truths have to be drawn from concepts rather than from calculations (arithmetic expressions)].

GAUSS explicitly pointed out that, in mathematics, *it can sometimes be useful not to overload claims of existence with demands for computability or definability*.

In contrast to the Lagrangeian proof, the Gaussian proof is not based on calculations, but on *concepts* - new concepts that GAUSS had coined, namely, the *concept* of congruence, the concept of the (algebraic) 'field' $\mathbb{Z}/(p)$ and, especially, the *concept* of the 'multiplicative inverse modulo p'. For the *'inverse'* a^* of an element a (modulo p), no formula (i.e., no

[5] RICHARD DEDEKIND first spoke of 'fields' in 1871. Without using this term, fields first appeared in GAUSS (1801), EVARISTE GALOIS (1830) and LEOPOLD KRONECKER (circa 1850). GAUSS, however, introduced the concept of congruence of two numbers as early as 1801, and established the laws of arithmetic that underlie calculating in congruences.

arithmetic expression, notation) can be given (which is the same for all prime numbers p and all numbers a with $1 \le a < p$) that is constructed by using only the ordinary addition and multiplication of integers. It can only be proven that a^* *must be there* among the numbers between 1 and $p-1$ and is uniquely determined.

The Gaussian statement made a deep impression on many mathematicians in the 19th century. PETER-GUSTAV LEJEUNE-DIRICHLET, in 1852, in his *'Remembrance speech for* CARL GUSTAV JACOB JACOBI, pointed to

> *"(...) die immer mehr hervortretende Tendenz der neueren Analysis (...) Gedanken an die Stelle der Rechnungen zu setzen (...)*
>
> [(...) the increasingly prominent tendency in recent analysis (...) to substitute thoughts (i.e. notions) for calculations (...)]

(G. LEJEUNE-DIRICHLET*'s 'Werke'*, volume 2, pp. 227-252, there p. 245). RICHARD DEDEKIND also quoted GAUSS's statement in 1895, in his report *'Über die Begründung der Idealtheorie'* (see DEDEKIND*'s 'Werke'*, vol. II, p. 54). At the end of his report *'Sur la Théorie des Nombres entiers algébriques'* (1877, *'Werke'* Vol. III, pp. 262-296), DEDEKIND emphasized the advantage of the conceptual approach to mathematics in his own words (in French) as follows:

> *"(...) une (...) théorie fondée sur le calcul, n'offrirait pas encore, ce me semble, la plus haut degré de perfection; il est préférable (...) de chercher à tirer les démonstrations, non plus du calcul, mais immédiatement des concepts fondamentaux caractérisques."*
>
> [A theory based on calculations would not yet, it seems to me, offer the highest degree of perfection. It is preferable to seek (to look for) proofs not based on calculations, but directly from fundamental, characteristic concepts.]

In his extensive report on *'Die Theorie der algebraischen Zahlkörper'* (Jahresbericht der DMV 4 (1897), pp. 175-546; reprinted in HILBERT*'s 'Gesammelte Abhandlungen'*, Volume 1, pp. 63-363, there p. 67), DAVID HILBERT, reminded of GAUSS's principle, writes

> *"... demzufolge man Beweise nicht durch Rechnung, sondern lediglich durch Gedanken zwingen soll.*
>
> [(...) that proofs should not be forced by calculation but only by thought.]

HERMANN MINKOWSKI expressed himself rather similarly:

> *"mit einem Minimum an blinder Rechnung, (und) einem Maximum an sehenden Gedanken die Probleme zu (be)zwingen."*
>
> [to master the problems with a minimum of blind calculation, (and) a maximum of seeing thoughts.]
>
> MINKOWSKI, in his paper on *'Peter Gustav Lejeune-Dirichlet und seine Bedeutung für die heutige Mathematik'* (Jahresberichte der DMV, vol. 14 (1905)), pp. 149-163, here p. 163).

These quotations show that, in the course of the 19th century, it gradually became common practice to give preference to conceptual argumentation over concrete calculations and, in particular, in theorems of existence, to show the existence of the objects

in question only in the sense of *'pure existence',* and not in the sense of constructing *'tangible examples'*.

But, it must also be mentioned that this view was not even close to being consensus among all mathematicians. Especially when, around the turn of the century 1897-1906, various antinomies (especially in the context of the young theory of sets) became known, everyone wanted to be careful and began again to be very sceptical about non-constructive proofs of existence. We will now discuss a typical example of this scepticism.

17.5 The Hilbertian basis theorem

In the *theory of algebraic invariants*, one studies the invariants of the *n-ary* forms, i.e., those algebraic expressions that can be constructed from the coefficients of the *n-ary* forms[6] that remain invariant under linear transformations of the coordinates.

For binary forms (i.e., $n = 2$) of degree ≤ 4, ARTHUR CAYLEY was able to show, in 1856, the existence of a finite basis with a constructive proof. (See P. GORDAN: CRELLE's Journal für die reine und angewandte Math., Volume 69 (1868), pp. 323-354.)

In the year 1868, PAUL GORDAN (1837-1912) had proved more generally, but again through an explicit construction, the *existence of a finite basis* for the class of all binary forms. "*There is*", hence, a finite system of invariants in terms of which all other invariants can be expressed rationally and integrally.

GORDAN's proof was essentially computational, using the structure of the elementary operations by which invariants can be generated. GORDAN had asked the question as to whether an analogous fact is also true for *n-ary* forms, where $n \geq 3$. The question remained unanswered for 20 years. HILBERT was able to answer it positively in 1888/1890. He proved the following theorem (see HILBERT*'s 'Gesammelte Abhandlungen'*, Volume 2 (Berlin 1933), pp. 199-257, Section V):

Theorem (D. HILBERT, 1890) *If* $n \geq 2$ *is a natural number and* S is *any system of n-ary forms (of any degree), then the system of all invariants of* S *has a finite basis.*

HILBERT*,* however, did not give an effective construction of a finite basis, but only showed that the assumption of non-existence would lead to contradictions. HILBERT*'s* proof rests on a simple lemma, which, in today's terminology, reads as follows[7]: *If* M *is a noetherian* R-modul*, then* $M[x_1, ..., x_n]$ *is a noetherian* $R[x_1, ..., x_n]$*-modul.*

HILBERT'*s* doctor-father FERDINAND LINDEMANN called this proof of existence *"unheimlich"* (*'shuddering', 'shivering'*), and PAUL GORDAN announced, in a loud voice (quoted after OTTO BLUMENTHAL: '*Lebensgeschichte*' in HILBERT*'s 'Gesammelte(n) Abhandlungen',* Volume 3 (Berlin 1935), pp. 388-429, there p. 394):

> *"Das ist keine Mathematik; das ist Theologie."*
> [That's not mathematics; that's theology.]

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[6] *n-ary* forms are polynomials in n variables $x_1, ..., x_n$ with coefficients from a commutative field K, in which all summands have the same exponent sum.

[7] A proof can be found in VAN DER WAERDEN: *'Algebra II'* (Berlin 1959), pp. 59-61, or EMIL ARTIN: *'Algebraische Geometrie'*, Lecture Hamburg 1962, page 5.

A little later, GORDAN softened his words a bit, and said:

"Ich habe mich davon überzeugt, dass die Theologie auch nützlich sein kann."
[I have convinced myself that theology can also be useful.]

But there were also many other mathematicians who could not be convinced. OSKAR BECKER (1889-1964), for example, reacted very violently, calling HILBERT's proof of the basis theorem a

"Schleichweg einer Schein-Konstruktion."
[... creeping path of a fake construction.]
O. BECKER: '*Mathematische Existenz*', 1927, op. cit., p. 471.

HERMANN WEYL (1885-1955) wrote, in his book '*Philosophie der Mathematik und Naturwissenschaften'* (München 1927, page 41; and still in the 3rd edition, 1966, p. 72):

"An den vielen Existenztheoremen der Mathematik ist jeweils nicht das Theorem das Wertvolle, sondern die im Beweis geführte Konstruktion; ohne sie ist der Satz ein leerer Schatten."
[In the many existence theorems of mathematics, it is not the theorem that is valuable, but the construction carried out in the proof; without it, the proposition is an empty shadow.]

For WEYL, the HILBERT basis theorem is therefore *'an empty shadow'*. But, in his book *'Classical Groups'* (Princeton 1939, p. 251), he nevertheless called HILBERT*'s* basis theorem the *"most important in the whole of algebra"*. One of these two statements can probably not be taken very seriously.

It is undisputed that proofs of theorems in which the existence of certain objects is shown by constructing them can be very valuable. But, existence statements in which objects asserted to exist are not specified, are by no means worthless. Should they and everything that follows from them be dispensed with in mathematics? The following example shows that existence theorems in which the existence of the things in question is only understood in an non-constructive way can also be of the greatest value.

17.6 Fast primality tests

The following theorem, in a slightly modified form, can already be found in the 7th book of EUCLID's *'Elements'* (there § 31):

Fundamental theorem of arithmetic: *Every natural number* $n \geq 2$ *can be represented as a product of prime numbers.*

EUCLID called a natural number n a *prime number* if there is no number z with $1 < z < n$ dividing n. If n is an arbitrarily given natural number, then the task of checking whether n has proper divisors seems to be very easy to perform - in principle. One only has to go through all numbers z between 2 and $\sqrt{n}$ and check whether z divides n. But this procedure can be carried out in practice only if n is quite small. Even with numbers that have about one hundred digits in the decimal system, the procedure is generally impracticable. Even if one could calculate very fast, i.e., if one could do about 10^{12} divisions per second, and had time to calculate for 10^{10} years (which is the estimated age of the earth), one would not

necessarily find a proper divisor.[8]

However, there is the famous characterization of primality credited to PIERRE FERMAT that is more suitable for primality tests than EUCLID's definition. It is the following:

Theorem (FERMAT, 1640): *For any natural number* $n \geq 2$, *the following is true*[9]:
n is a prime number $\Leftrightarrow \quad \forall a \in \mathbb{N}(2 \leq a < n \implies a^{n-1} \equiv 1 \ (modulo\ n))$.

The theorem can be used for a reliable and secure primality test. If n is any natural number, $n \geq 2$, then one has to calculate, for each number a (for $2 \leq a < n$), its power a^{n-1} and check whether $a^{n-1} - 1$ is a multiple of n or not. If this is not the case for all a, then n is composite, but if it is always the case, then n is a prime-number. The test is rather quick, since the powers are easily calculated when n is written down in its 2-adic representation. (Notice that there are various modifications of the FERMAT-test that are significantly faster, but, for our purpose here, the above-mentioned version suffices.)

If, for a given number n, the FERMAT-test is negative, in the sense that n is signified as a composite number, then the test will not provide us with a factorization. Here, the existence of factors is understood in the non-constructive sense as *'pure existence'*. But, as we have seen in the previous paragraphs, for most composite numbers $n > 10^{100}$, it is, in general, impossible to find numbers a and b such that $2 \leq a < n \ \ \& \ \ 2 \leq b < n \ \ \& \ \ n = a \cdot b$.

The FERMAT test will only give an answer to the question as to whether n is composite or not; the test will not, in addition, provide us with a factorization in case the answer is negative. If one simply wants to find divisors very concretely, one has to calculate for an extremely long time. *This example makes the difference between the constructive and the non-constructive conceptions of existence in mathematics quite clear, and also shows the great usefulness of the non-constructive conception.*

The concept of constructibility can also be defined more broadly, for example, by ignoring the required time or the required memory space, as is done in CHURCH's thesis, which identifies the concept of a computable function with the concept of a recursive function. But this is an *idealization* of the concept of computability, and this idealized concept has, as its content, only that which is *per se* computable. In order to justify the claim that the idealized conception correctly represents the content of the concept of (real) computability, one would have to imagine an immortal spirit (or an 'immortal mathematician', as BECKER expressed it, op. cit., p. 669) who can calculate for as long as he likes without taking into account the time and memory required. Even that which sometimes seems to be so concrete and constructive is based on idealizations .

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[8] If n is a natural number with about hundred digits in the usual decimal system, then, according to the classical prime-number-theorem (J. HADAMARD & CH. DE LA VALLÉE-POUSSIN, 1896), one would have to test about $\sqrt{n}/\ln(\sqrt{n}) \approx 8 \cdot 10^{47}$ prime numbers to see if they are divisors of n. Even an electronic computer that can perform 10^{12} divisions per second (i.e., $3 \cdot 10^{19}$ divisions per year) would generally take at least 10^{28} years.

[9] The proof of '$\Longrightarrow$' is well known. Proof of '$\Longleftarrow$': Assume $n = ab$ with $2 \leq a < n$ & $2 \leq b < n$ & $a^{n-1} \equiv 1 \pmod{n}$ & $b^{n-1} \equiv 1 \pmod{n}$. Then, $0 \not\equiv a = a \cdot 1 \equiv a \cdot b^{n-1} = (a \cdot b) \cdot b^{n-2} \equiv 0 \pmod{n}$ - a contradiction!

17.7 Discussion

'To even try to prove that something is there, is a difficult thing.'
S. KIERKEGAARD: *'Philosophiske Smuler'* (1844), Chapt. III.

What is the meaning of the concept of *'existence'* in colloquial language and what should it mean in mathematics and in mathematical logic?

In Latin, the word '*sistere'* means 'to place', 'to set up' and the preposition '*ex'* means 'out'. '*Existere'* therefore means 'to stand out', 'to emerge',.... *'Existentia'* is literally that which emerges from the totality of all that which is at hand or in stock. In colloquial language, *'existence'* is used synonymously with *'being in presence'* (i.e., to lie in front of the hands, to be palpable with the hands). Also, the manner in which *'it exists'* is used is synonymous with *'it is given'*, since *'it is given'* expresses the impersonal *'to be given'* (as in French: *il y a*). It is important to note that, in colloquial language, it is left open as to how the object is given; but it must be 'given' somehow.

In mathematics, however, the word *'existence'* is used in a metaphorical sense.[10] The word cannot be used without certain restrictions. The use of concepts in mathematics always has to follow rules that are precisely formulated, but these rules have to be determined and, in fact, it took quite some time to determine these rules.

From antiquity on up to the 19th century, proofs of existence were mostly constructive. It is, however, noteworthy that ARISTOTLE treated *'existence'* in the sense of *'pure existence'*. This is clear from his treatment of the syllogisms in his *'Analytica priora'*, Chapter IV, 26b15, where, for example, it is stated that an expression of the type *'not every C applies to B'* is equivalent to an expression of the type *'some C does not apply to B'* (cf. JACQUES BRUNSCHWIG, op. cit.). This means, in modern terminology, that existential quantification $\exists x$ is equivalent to the expression $\neg\forall x\neg$. Since then, other logicians have repeated this equivalence, including GOTTLOB FREGE, in his *'Begriffsschrift'* from 1879, p. 23.

The equivalence mentioned in the previous paragraph is not valid in colloquial language, but it is valid in 'Formal Logic', since formalized logic is purified from any ontological commitment. Here, 'existence' can only be understood in the sense of pure existence, i.e., somewhat loosely expressed, as 'existence' for logical reasons, or, stated differently, 'existence without construction'. It turned out that, together with the classical rules of 'negation' (e.g., the laws of OCKHAM-LEIBNIZ-DEMORGAN), in which a certain duality between 'conjunction' and 'disjunction' is expressed, the *'universal quantifier'* $\forall$ and the *'existential quantifier'* $\exists$ in its purified version are 'dual quantifiers'.

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[10] For DAVID HILBERT (1899, 1900, in his lecture on *'Mathematical Problems'*) and HENRI POINCARÉ (1902), a mathematical object exists in a mathematical theory when the theory, in which the existence of such an object is assumed, is consistent, i.e., is free from contradictions. This, however, only means that, in some models of the theory, such objects really exist. Additionally, this kind of 'existence' is 'pure existence', i.e., 'existence' without construction. Compare PAUL BERNAYS, op. cit., and JACQUELINE BONIFACE, op. cit.

It is undisputed that a constructive proof of existence can provide more information than a non-constructive proof of existence, - if it can be achieved at all. But it is also worthwhile to distinguish between 'pure existence' and 'effective construction or calculation'. DAVID HILBERT argued, in his lecture in Hamburg in 1927, as follows:

> *"Das Wertvolle der reinen Existenzbeweise besteht gerade darin, daß durch sie die einzelne Konstruktion eliminiert wird, ... so daß allein das für den Beweis Wesentliche deutlich hervortritt."*
>
> ['The value of pure existence proofs consists precisely in that the individual constructions have been eliminated ... and only that which is essential to the proof stands out clearly'.]
>
> D. HILBERT: *'Die Grundlagen der Mathematik'*, op. cit., p. 79.

It is therefore reasonable to develop the constructive aspects of mathematical existence in a separate 'constructive mathematics', as far as this is possible (see the monographs of MICHAEL BEESON, ERRETT BISHOP, CHARLES CHIHARA, op. cit. and the essays of DIRK VAN DALEN, op. cit.)

In the lecture that was scheduled for the 'International Congress of Mathematicians' in Chicago in 1893, DAVID HILBERT wrote (cf. HILBERT*'s 'Gesammelte Abhandlungen',* Volume II, p. 383):

> *"In der Geschichte einer mathematischen Theorie lassen sich meist 3 Entwicklungsperioden leicht und deutlich unterscheiden. Die naive, die formale und die kritische. Was die Theorie der algebraischen Invarianten anbetrifft, so sind die ersten Begründer derselben CAYLEY und SYLVESTER, zugleich auch als die Vertreter der naiven Periode anzusehen: an der Aufstellung der einfachsten Invariantenbildungen und an den eleganten Anwendungen auf die Auflösung der Gleichungen der ersten 4 Grade hatten sie die unmittelbare Freude der ersten Entdeckung. Die Erfinder und Vervollkommner der symbolischen Rechnung CLEBSCH und GORDAN sind die Vertreter der zweiten Periode, während die kritische Periode in den ... Sätzen* (die HILBERT in seinem eigenen Vortrag nannte) *ihren Ausdruck findet."*
>
> [In the history of a mathematical theory usually three developmental stages can be easily and clearly distinguished. The naive, the formal and the critical. As for the theory of algebraic invariants, the first founders of it, CAYLEY and SYLVESTER, are to be regarded as the representatives of the naive period: they had the immediate joy of first discovery in the drawing up of the simplest formations of invariants and in the elegant applications to the solution of equations of the first four degrees. The inventors and perfecters of the symbolic calculus CLEBSCH and GORDAN are the champions of the second period, while the critical period is expressed in the ... theorems (which HILBERT mentioned in his own lecture).]

HILBERT spoke of the *'critical period of development'*. and referred to KANT (without quoting him). Just as KANT, in his *'Kritik der reinen Vernunft'* (see Chapter 12), wanted to explore *the possibilities and limits* of pure reason, so, according to HILBERT, in the development of a mathematical theory, the *'critical period'* is reached as soon as it is a matter of exploring its *possibilities and limits* in full generality.

For invariant theory, HILBERT himself indicated the three periods. In the first period, individual problems are solved *ad hoc* through calculation. In the second period, the

individual discoveries made in the first period are further developed into a general *theory* and the problems arising in it are solved systematically and explicitly by specifying symbolic expressions (formulae, terms). In the third period, on the other hand, the *possibilities and limits* of the theory constructed in the second step are examined, whereby, often enough, it can only be shown that it behaves this way or that way *'in principle'*. We will return to this in Chapter 20.

In geometry, the period from THALES to the Pythagoreans could be called the first period, and the period beginning with EUCLID and extending to LEGENDRE could be called the second period. The third period, beginning with GAUSS, BOLYAI and LOBATSCHEFSKIJ, which began with the investigations of the parallel postulate, could then be considered the *'critical period'*. HILBERT himself wrote, in the closing words of his book on *'The Foundations of Geometry'* (1899), that he wanted to make *'a critical study of the principles of geometry'*, indicating that he considered his work to be in the third period.

The appropriate concept of *'mathematical existence'* in the *'naive period'* may, perhaps, be the constructive one. In the *'formal period'*, it may, perhaps, already be the non-constructive one. However, in the *'critical period'* of a mathematical theory, it is, doubtless, the purified non-constructive view that, in general, is the appropriate one.

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Chapter 18
The formal and the contentual Position

> *"In Arithmetic ... we regard not the things, but the signs."*
>
> GEORGE BERKELEY: *'Treatise concerning the Principles of Human Knowledge',* Dublin 1710, §122.
>
> *"Il n'est pas vrai, qu'en Algèbre on raisonne sur les signes: on raisonne toujours sur les idées qu'il représentent."* [It is not true, that in Algebra one reasons on the signs; one always thinks about the ideas they represent.]
>
> LOUIS COUTURAT: *'Les principes des Mathématiques', Paris 1905, p. 276.*

A formalist, in the colloquial sense of the word, is someone who follows the prescribed formalities exactly and does not question them. Formalists are therefore not well-liked in all areas of life. They are accused of neglecting, in favor of *formalities*, the *contents*, which should be the final issue. It is obviously also assumed that *form* and *content* do not always coincide.

ANTOINE ARNAULD (1612-1694) and PIERRE NICOLE (1625-1695) first spoke of the *content of a concept* (*compréhension de l'idée, comprehensio ideae, Begriffsinhalt*) in their book *'La Logique ou L'Art de penser'* (Paris, 1662), also called *'La Logique de Port Royal'.* By this, they meant *'the totality of all the attributes which the concept includes and which cannot be removed without destroying the concept'*. In this sense, we also want to speak of the *'content'* of a concept.

In mathematics, it is a virtue to adhere strictly to the prescribed formalities. However, the fundamental question arises as to whether, in the foundation of a mathematical theory, the formalisms serve to **reproduce** the given content as accurately and clearly as possible or, vice versa, whether they serve to **introduce** the content.

Our aim in this chapter is to clarify this question. Behind that question is the great old problem of *whether the objects to be studied in mathematics are somehow predetermined for us (with all their attributes and their essential properties), or whether we ourselves can only introduce the objects as fictions by simply postulating with the help of axioms (or postulates) the properties they should have.*

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Science Networks. Historical Studies 62, https://doi.org/10.1007/978-3-031-27304-9_18

18.1 The contentual and the formal point of view

So, the problem in construcing mathematical theories or disciplines is

- (A) whether we may rely on the *contents* of the basic concepts, assuming that they are predetermined and already known to us, even if they cannot be completely formalized and *'can only be shown approximately in experience or in vivid imagination'* (*"die nur ungefähr in der Erfahrung oder in der anschaulichen Vorstellung aufweisbar sind",* cf. HILBERT-BERNAYS, op. cit., Vol. I, p. 7)

  - or -
- (B) whether we can rely only on concrete *formalisms*, even if they cannot fully grasp real facts, or perhaps are not based on *'experiences or illustrative ideas'* at all.

It is said that whoever advocates the first of the two alternatives takes the *'contentual position',* and whoever tends towards the second alternative takes the *'formal position'* (or the *'formalist position'*).

(A) When one takes the '**contentual standpoint**' with regard to a theory, one assumes that the contents of its basic concepts are well known to us and that it is not absolutely necessary - and can even be refused - to name and define them explicitly. The basic concepts are introduced *'by reference to known experiences'* and the principles of the theory are presented *'either as evident facts ... which one can make clear to oneself'*, or are declared *'as an extract of experience complexes'* (cf. HILBERT-BERNAYS, op. cit., Vol. I, p. 2). Even if one is prepared to formalize the foundations of the theory, at least in part, one assumes *that the theory deals with contents that precede any formalization and are superior to any formalization.* Additionally, one may always refer to these contents in proofs without making them explicit.

Euclidean geometry, for example, from antiquity until the beginning of the 19th century, has been constructed from a contentual point of view. Although several basic concepts such as *'point'*, *'straight line'*, etc., could not be defined satisfactorily, most readers of EUCLID's *'Elements'* had been convinced that all of these concepts are nevertheless well understood, and hence are acceptable in mathematical reasoning. The renunciation by DESCARTES, PASCAL, ARNAULD and others of precise definitions of the basic concepts in geometry (cf. Chapter 10, section 10.2) makes it totally clear that, in such a geometry, the contentual standpoint is taken.

The same is the case in 'naive' set theory, in which EMIL BOREL, FELIX HAUSDORFF and others had been convinced that the whole content of the concept of a *'set'* is given to us by intuition and need not be defined explicitly (cf. Chapter 10, section 10.2).

(B) If one adopts the '**formal standpoint**' with regard to a theory, then one assumes *that the contents that the theory is supposed to deal with are only introduced exactly and precisely through the specification of formalisms (calculi, systems of rules, axiom systems), and are thus subordinate to the formalisms.*

The formal point of view is thus characterized by the fact that only that which can be represented in a formal system is acceptable as the basis of a mathematical theory and that everything can be deliberately ignored that cannot be treated with formalisms (calculi, axiom systems, systems of rules). It is therefore a position that wants to leave out

everything that is perhaps often considered, but cannot be formalized. *What cannot be formalized cannot be treated with the means of mathematics.*

The axioms here are, in principle, arbitrary settlements; whether they are *true or not* is not at issue. Therefore, nothing is said about the *truth* of the theorems; it is required only that they are *provable.*

An example is the theory of imaginary and complex numbers. After GERONIMO CARDANO (1501-1576) first came across so-called *'imaginary numbers'*, such as $i = \sqrt{-1}$, $\sqrt{-9}$ and $\sqrt{-15}$, which are neither positive nor negative, but *'something third of a hidden nature'* ("*quaedam tertia natura abscondita*"), and which he therefore rejected from a contentual point of view, RAFAEL BOMBELLI (1526 - ca.1572/1573), in his book *'L'Algebra Opera'* (1572, 1579), introduced and rehabilitated them from a formal point of view. For BOMBELLI, negative numbers are (positive) numbers preceded by the word *'meno'* (for 'minus', or 'less than') and imaginary numbers are positive or negative numbers preceded by the word *'di meno'* ('even less'). He does not derive the calculation rules from the content of these prefixed words. In this respect, the signs *' – '* and *'i'* are not treated as *symbols,* but as *'empty signs'* (cf. section 18.2). What is meant by them should only be clear from the required rules of calculation. BOMBELLI gives four rules for the multiplication of positive and negative numbers, for example:

"meno via meno fa più", etc,	[minus times mines is plus], etc,
.....	.....

and eight rules for multiplying real numbers by imaginary numbers, e.g.:

"più die meno via più di meno, fa meno".	[*i* times *i* is minus],
"meno via meno di meno, fa più di meno".	[minus times -*i* is +*i*],
"meno di meno via più di meno, fa più"	[-*i* times *i* is plus], etc.
.....	....

Thus, no extra-mathematical sense is given to the imaginary numbers and the way in which they can be multiplied with each other. That which they mean is contained exclusively in the system of rules of calculation.

Calculating with complex numbers has many amazing consequences. For example, if $p > 0$ and $q > 0$ are both positive real numbers and $(p/3)^3 > (q/2)^2$, then all three solutions to the cubic equation $x^3 = px + q$ are real and the following term

$$\sqrt[3]{\frac{q}{2}+\sqrt{\frac{q^2}{4}-\frac{p^3}{27}}}+\sqrt[3]{\frac{q}{2}-\sqrt{\frac{q^2}{4}-\frac{p^3}{27}}}$$

describes one of the three real solutions. But, in this description, two imaginary numbers appear, an appearance that cannot be avoided. This is why it is called *casus irreducibilis*. We see that roots of negative numbers are also very useful and significant.

Another convincing example for the adoption of the formal standpoint is the introduction of *'ideal numbers'* in algebraic number theory. It was ERNST EDUARD KUMMER (1810-1893) who, in the years 1844-1847, op. cit., observed that, in several algebraic rings of numbers, e.g., in $\mathbb{Z}[\sqrt{-5}]$, the decomposition of numbers into irreducible numbers is not

always unique, and hence irreducible numbers are not necessarily prime-numbers, e.g.:

$$6 = 2 \cdot 3 = \left(1 + \sqrt{-5}\right) \cdot \left(1 - \sqrt{-5}\right),$$

where the irreducible number 2 divides 6, and hence also $\left(1 + \sqrt{-5}\right) \cdot \left(1 - \sqrt{-5}\right)$, but 2 divides neither $\left(1 + \sqrt{-5}\right)$ nor $\left(1 - \sqrt{-5}\right)$. In order to recapture unique factorization, KUMMER had the brilliant idea to introduce a new kind of number, which can be viewed as the greatest common divisor without belonging to the field of fractions of the algebraic number ring. He called the elements belonging to that field the *'actual'* numbers, and the new objects not belonging to that field *'ideal numbers'*.

KUMMER introduced these *'ideal numbers'* from a formal point of view by introducing *signs* as names for them, as well as defining the algebraic operations of addition, multiplication and divisibility for them. These signs do not have any content *per se*; they are *'empty signs'*. But the laws of the algebraic operations to which these signs are subjected attach the desired meanings to them. This indicates the adoption of the formal point of view.

Notice that KUMMER did not construct objects that are *'ideal numbers'* per se. It was RICHARD DEDEKIND who, in 1871 and the following decades, was able to generalize KUMMER's theory. DEDEKIND's theory, however, is based on *objects*, which he called *'ideals'*. They are set theoretically-defined objects whose algebraic properties are somehow encoded in them. Set theory was not yet disposable for KUMMER.

18.2 '**Symbols**' **and** '**empty signs**'

The characters, letters or signs used in the vocabulary of a mathematical theory, if they are not *variables*, can be either *symbols* or so-called *'empty signs'*. We will explain the difference between the two types of constants.

The Greek word *'symbolon'* (σύμβολον) is composed of the words *'syn'* (σύν = 'together') and *'bállein'* (βάλλειν = 'to throw'). Hence, *'symballein'* means *'to join',' to put together'*. Originally the word *'symbolon'* (*'symbol'*) referred to the fragments of an object broken into pieces to be used as marks of identity. The various pieces of the broken object had to be handed out to the members of a group of persons and the members could prove their membership by showing that their piece fits into the composition of the pieces of the others provided the broken object can be completely reconstructed in this way.

In later times the term *'symbol'* was used more generally to denote signs to with special meanings had been appended (adjoined, or attached). One had to be initiated into the symbolism in order to be able to understand the appended meaning.

In Mathematics a sign is called a *'symbol'* if it is accompanied by meanings that cannot be logically derived from the axioms or rules of the corresponding theory.

A sign is an *'empty sign'* if no (additional) meaning is appended to it, or, more precisely, if only those meanings are attached to it that are granted by the underlying system of rules, postulates or axioms.

The *'emptiness'* of signs guarantees that in proofs nothing else is used other than those bits of information that are explicitly stated in the system of rules, postulates or axioms,

upon which the whole proof rests. This is necessary to know when the correctness of a proof is doubted (cf. also Chapter 19, section 19.8).

The *'emptiness'* of signs also allows us to interpret them in ways different from their usual interpretations, and makes it possible to prove claims of independence. This is the case, e.g., in the construction of models of non-Euclidean geometry, in which *'straight lines'* are interpreted by certain curved lines, thus showing that the empirical notion of *'straightness'* is undefinable and the Euclidean postulate of parallels is unprovable - under the assumption that geometry is understood as a formal theory. - Empty signs were already used by JOHANN HEINRICH LAMBERT (1728-1777) in 1766 in his investigations into the Euclidean parallel axiom - cf. PAUL STÄCKEL, 1895, op. cit., p. 162.

In literature the word 'symbol' is rarely used correctly. We therefore emphasize once again that we always intend to use it in the original literal sense (as explained above).

18.3 The dispute over the introduction of negative numbers

The introduction of *'negative'* and *'imaginary numbers'* in the 15th and 16th centuries (see above), as well as the introduction of *'ideal numbers'* in algebraic number theory by ERNST EDUARD KUMMER in 1847, gave the first impulses that led to the emergence of the *formal standpoint.* But the discovery of the *duality principle* of projective geometry by JOSEPH DIEZ GERGONNE (1771-1859) in 1826, and the construction of models of non-Euclidean geometries (BERNHARD RIEMANN 1854/1867, EUGENIO BELTRAMI 1868, FELIX KLEIN 1870, DAVID HILBERT 1899 and others), also contributed significantly to this. All of these discoveries led to the fact that formal arguments began to prevail over contentual arguments. In the case of the principle of duality [1], for example, it became apparent that the proofs of the validity of certain geometric statements do not depend on *contentual* or intuitive provisions of the basic concepts, but only on extrinsic *formal* conditions.

The greatest influence on the development of the formalist position, however, was the dispute over the definition of the essence of mathematics as a *theory of quantities* (*'science des quantités', 'Größenlehre'*), which has been valid since antiquity. This dispute began as early as the 16th century, after NICOLAS CHUQUET (1484), MICHAEL STIFEL (1544) and others gave the first systematic representations of calculating with *positive* and *negative* numbers. The dispute intensified after the introduction of *imaginary* and *complex* numbers by RAFAEL BOMBELLI (1572/1579), LEONHARD EULER (1749) and others.

MICHAEL STIFEL did not understand the 'negative' numbers as true, but rather as *fictitious* numbers. In his *'Arithmetica integra'* (Nürnberg 1544, p. 249r), he called them *"numeri ficti infra nihil"* (*'fictitious numbers smaller than nothing'*). [2] ISAAC NEWTON accepted this description and wrote, in his *'Arithmetica Universalis'* (London, 1707, p. 3, op. cit.):

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[1] In the case of projective planes, e.g., the *principle of duality* states that any theorem about projective planes remains true, when the words 'points' and 'lines' are interchanged within it. In the case of *n-dimensional* projective spaces $\Pi(n)$, the *principle of duality* is a bit more complicated to state, cf. GÜNTER PICKERT: *'Analytische Geometrie',* Akad. Verlagsgesellschaft Leipzig 1961, pp. 314-315.

[2] For more details on *'fictitious objects'* in mathematics, see U. FELGNER: *'Anmerkungen zum Begriff des fiktiven Gegenstandes in der Mathematik',* 2014, op. cit.

"Quantitates vel Affirmativae sunt seu majores nihilo, vel Negativae seu nihilo minores." In the English translation of his book (London, 1728, p.3), we read: *"Quantities are either Affirmative, or greater than nothing; or Negative, or less than nothing."*

Even EULER described the 'negative numbers', in the first part of his book *'Vollständige Anleitung zur Algebra'* (1770, Part I, Sect. 1, Chapter 1, § 18), as numbers that were *'less than nothing'* (*"weniger als nichts"*). BOMBELLI had already called the imaginary numbers *"di meno",* i.e., *'even less than nothing'.*

But, at that time, 'numbers' were still understood as *quantities* (and hence as symbols), and so the question soon arose as to whether quantities could be *'less than nothing'* at all. If so, then they are also 'nothing', and why should one 'nothing' be smaller than another 'nothing'?

FRANCIS MASÈRE (1731-1824), in his *'Dissertation on the Use of the Negative Sign in Algebra'* (Cambridge, 1758), therefore disapproved of the use of 'negative numbers' in mathematics. In the town in which NEWTON had lived, this caused a sensation. The Irish-English novelist LAURENCE STERNE (1713-1768), who had studied in Cambridge and who had certainly heard about the controversy concerning negative numbers, made an ironic allusion to it in his novel *'Tristram Shandy"* (Chapter 19, written in the same year, 1758):

"Jack, Dick, and Tom (...) my father called neutral names. (The name) *'Andrew' was something like a negative quantity in Algebra with him, (...) twas worse, he said, than nothing. But of all the names in the universe, he had the most unconquerable aversion for TRISTRAM."*

Doubts concerning the reasonability of negative numbers also spread over continental Europe. In the '*Encyclopédie méthodique*' by DIDEROT and D'ALEMBERT (Paris, 1751-1772), this question is discussed. It says there (cf. keyword 'Négatif'):

"Dire que la quantité negative est au dessous du rien, c'est avancer une chose qui ne se peut pas consevoir."

[To say that a negative quantity is below nothing is to advance something that you can't conceive of.]

FRANCIS MASÈRES considered the theory of negative numbers to be

„... obscure and difficult, and disgusting to men of a just taste for accurate reasoning"

F. MASÈRES: ‚*Tracts on the resolution of affected algebraick equations*', London 1800, p. LV, and p. LIX und S. 337–343.

AUGUSTUS DE MORGAN (1806–1871) wrote, in his book *'On the study and difficulties of Mathematics*' (1831, chapter IX, pp. 103–104):

*"*3 – 8 *is an impossibility*,"

because numbers stand for *quantities* and (for him) the subtraction stroke is a *symbol* that has the meaning of *taking away.* But you can only take something away where there is already something there. This, however, is a contentual argument and not a formal one.

The 'negative numbers' were also called *'privative numbers'*, as they arise from the positive (i.e., affirmative) numbers by prefixing the subtraction stroke. (In Latin: privare = to take away, to separate, detach, abstract, secluse,....)

IMMANUEL KANT was probably the first to point out, in his essay *'Versuch den Begriff der negativen Größen in die Weltweisheit einzuführen'* (1763, introduction to the 1st section), that, in mathematics, one should not always refer to the *contents* of definitions given in colloquial language, but should only follow the rules for the use of otherwise undefined terms. He wrote about the introduction of negative numbers:

> *"Der Begriff der negativen Größen ist in der Mathematik lange in Gebrauch gewesen ... . Indessen ist die Vorstellung, die sich die Mehrsten davon machen, und die Erläuterung, die sie gaben, wunderlich und widersprechend, obgleich daraus auf die Anwendung keine Unrichtigkeit abfloß; denn die besonderen Regeln vertraten die Stelle der Definition und versicherten den Gebrauch, was aber in dem Urteil über die Natur dieses abstrakten Begriffs geirrt sein mochte, blieb müßig und hatte keine Folgen."*
>
> [The concept of negative quantities has long been in use in mathematics .... But the conception most people have of it and the explanation they have given is peculiar and contradictory, although no inaccuracy has come out of it; for the special rules took the place of the definition and ensured its use, but whatever might have been mistaken in the judgment about the nature of this abstract concept remained idle and had no consequences.]

KANT's statement contains the cautiously formulated hint that it might be advisable to introduce negative numbers from a formal point of view, and not from a contentual one (as BOMBELLI had already done in the 16th century with respect to imaginary numbers, and other mathematicians, in later times, have done it as well, e.g., KUMMER, HANKEL,...).

But it was not until half a century later (approximately from 1820/1830 on) that a reform began in mathematics, characterised by a gradual emergence of the formal standpoint. The reform was initiated in England by CHARLES BABBAGE (1792-1871) and GEORGE PEACOCK (1791-1858), and continued in Germany by MARTIN OHM (1792-1872), HERMANN GRASSMANN (1809-1877), ERNST EDUARD KUMMER (1810-1893), HERMANN HANKEL (1839-1873), MORITZ PASCH (1843-1930), DAVID HILBERT (1862-1943) and others (cf. E. KNOBLOCH, J.M. DUBBEY and H. PYCIOR, op. cit.). They began (only then!) to understand that mathematics was no longer a *theory of quantities*, and this had the consequence that mathematical objects no longer had to be (positive) 'quantities' and arithmetic and algebraic operations no longer had to be interpreted as dealing with (positive) 'quantities'. The operations +, –, · and : (addition, subtraction, multiplication and division) were no longer bound to the colloquial meanings of 'adding', 'taking away', 'multiplying' and 'dividing into parts of equal size'; they could also be interpretable in an unconventional way. This reform led to a new understanding of *'algebra'*.

From then on, algebra was no longer limited to the study of the laws of calculating with natural and with real numbers, but could devote itself to more general areas of calculation (or *'structures',* such as groups, rings, fields, skew-fields, near-fields, ternary-fields, lattices, Boolean algebras, etc.).

GEORGE PEACOCK wrote, in his '*Treatise on Algebra*' (op. cit., 1830, p. 71, § 78):

> *„Algebra may be considered, in its most general form, as the science which treats of the*

combinations of arbitrary signs ... by means of defined though arbitrary laws: for we may assume any laws ... so long as our assumptions are ... not inconsistent with each other.“

DUNCAN FARQUHARSON GREGORY (1813-1844) wrote, in his essay '*On a difficulty in the Theory of Algebra*' (Cambridge Math. Journal 3 (1842), pp. 153-159), that a sign is algebraically defined

„... when its laws of combination are given“.

This view allowed GEORGE BOOLE (1815-1864) to present the laws of propositional logic within the framework of the new *formal algebra,* which led to what is known as '*Boolean algebra*'. BOOLE showed that logical reasoning could also be represented (and formalized) as a kind of algebraic calculation. In the preface to his book *'The Mathematical Analysis of Logic, being an essay towards a calculus of deductive reasoning'* (Cambridge, 1847), BOOLE wrote that the *formal algebra* developed by PEACOCK and others dealt with uninterpreted, empty signs, and that they could therefore also be interpreted in logic. BOOLE thus succeeded in establishing logical laws in the form of algebraic equations.

It is noteworthy that algebraic signs were no longer treated as *symbols*, i.e., as signs with a fixed (and unique) extramathematical interpretation, and hence that mathematics was no longer qualified as a *science of quantity*. In the course of time, other *contentual* provisions were also eliminated. An example will explain this.

At the beginning of the 19th century, *'groups'* were always substitution groups and the group operation was defined *contentually* by the successive execution of the individual substitutions. Only at the end of the 19th century was the operation introduced *purely formally* by so-called group tables and, alternatively, by generators and defining relations. The use of group tables goes back to ARTHUR CAYLEY (*'On the theory of groups',* Proceedings London Math. Soc. 9 (1878), 126-133, reprinted in CAYLEY*'s* Works, volume 10, pp. 324-330). He gave the following, purely formal definition of the concept of a 'group':

„A group is defined by means of the laws of combinations of its symbols.“

Another *purely formal* description of 'groups' by generators and the relations among them was introduced by WALTER DYCK, in his *'Gruppentheoretische Studien'* (Math. Annalen 20 (1882), pp. 1-44). We will come back to this in Chapter 19.

18.4 Combining the contentual and the formal standpoints

In Germany, MARTIN OHM and HERMANN GRASSMANN were probably the first to take up and to develop further the ideas of BABBAGE, PEACOCK and others. For them, pure mathematics was no longer the *'science of quantity'*, but a '*science built up from a formal point of view*'.[3] The standpoint taken here was soon called the *'formal standpoint'*.

But an important step beyond PEACOCK and the others was taken by HERMANN HANKEL, in his book *'Theorie der complexen Zahlensysteme'* (Leipzig, 1867). He adopted PEACOCK's

[3] OHM in the preface to the 2nd and 3rd edition of his *'Versuch eines vollkommen konsequenten Systems der Mathematik'*, 1822/1852, and GRASSMANN in his work *'Die lineale Ausdehnungslehre'* (Leipzig, 1844, p. 23).

formal point of view, but did not ignore the *contentual point of view*. He recommended that, in founding mathematical theories of natural numbers, of whole numbers, of rational numbers, of real numbers and of complex numbers, one should, in a first step, formalize as carefully and completely as possible all contentual aspects that are known to us from using these numbers and the operations with them in our daily life. This will lead us to formulate the general laws that govern the calculations with these numbers. In particular, HANKEL discussed the *content* of the meaning of the expressions 'addition', 'multiplication' and 'unit' from colloquial language (pp. 1-4), and finally arrived at a formulation of general laws for calculation with natural numbers (pp. 36 etc.). HANKEL wrote about this (op. cit., p. 2):

> *"Die hier angegebenen Eigenschaften der Addition sind ausreichend, um aus ihnen alle weiteren Folgerungen über Summenbildung abzuleiten, ohne daß man sich jemals dabei der realen Bedeutung der Addition erinnern müßte. Sie bilden insofern das System der Bedingungen, welche nötig und ausreichend sind, um die Operation formal zu definieren."*
>
> [The properties of addition given here are sufficient to derive from them all further conclusions about summation, without ever having to remember the real meaning of addition. In this respect they form the system of conditions which are necessary and sufficient to formally define the operation.]

The formal position has been formulated very clearly here. However, there is still one axiom missing from HANKEL's list of axioms of arithmetic, namely, the principle of complete induction. RICHARD DEDEKIND referred to this axiom in his essay *'Was sind und was sollen die Zahlen?* (1888) - cf. Chapter 19. Only with this highly important axiom (which is formulated in 2nd order logic) was the axiom system for the arithmetic of natural numbers complete.

HANKEL goes on to introduce the laws of calculation that are valid in the domains of whole numbers (we will discuss this in the next section 18.5), and of rational numbers, and so forth.

MORITZ PASCH (1843-1930) shared the views of HANKEL and outlined the foundation of geometry in a similar way in his *'Vorlesungen über neuere Geometrie'* (1882, op. cit.). In a first step, those concepts are sought out that can no longer be defined with more general concepts. These concepts are the *fundamental concepts,* which he also called *'Kernbegriffe'*. They are used to define the other concepts, and then to formulate the basic principles of geometry. In this way, the contentual and descriptive aspects of geometry are formulated, *"der empiristische Unterbau"* as he puts it (op. cit., p. 174). In a second step, a formal axiomatic theory of geometry is set up in which the fundamental concepts (the *'Kernbegriffe'*) appear as undefined terms, and are thus treated as *empty signs*. Finally, the basic principles are stated as *axioms* (‘*Kernsätze*’), and these axioms define the fundamental concepts in an implicit way (cf. op. cit., pp. 15-16, p. 40, p. 48). PASCH wrote about this procedure (op. cit., p. 90):

> *"Es muß in der Tat, wenn anders die Geometrie wirklich deduktiv sein will, die Deduktion überall unabhängig sein vom Sinn der geometrischen Begriffe, wie er unabhängig sein muß von den Figuren; nur die in den benutzten Sätzen und Definitionen niedergelegten Beziehungen zwischen den geometrischen Begriffen dürfen in Betracht kommen."*

> [Indeed, if otherwise geometry wants to be truly deductive, deduction must be everywhere independent of the meaning of the geometric concepts, as it must be independent of the figures; only the relations between the geometric concepts laid down in the propositions and definitions may be taken into consideration.]

Again, the formal position has been clearly stated here. However, the system of PASCH's axioms (*'Kernsätze'*) is not complete either; a complete axiom system was only established by DAVID HILBERT in 1899 - see Chapter 20.

DAVID HILBERT and PAUL BERNAYS also shared the views of HANKEL and PASCH. They wrote, in their work *'Grundlagen der Mathematik'*, vol. 1, op. cit., p. 2-3):

> *"Die formale Axiomatik bedarf der inhaltlichen notwendig als ihrer Ergänzung, weil durch diese überhaupt erst die Anleitung zur Auswahl der Formalismen und ferner für eine vorhandene formale Theorie auch erst die Anwendung auf ein Gebiet der Tatsächlichkeit gegeben wird. Andererseits können wir bei der inhaltlichen Axiomatik nicht stehenbleiben, weil wir es in der Wissenschaft, wenn nicht durchweg, so doch vorwiegend mit solchen Theorien zu tun haben, die gar nicht vollkommen den wirklichen Sachverhalt wiedergeben, sondern eine vereinfachende Idealisierung des Sachverhaltes darstellen und darin ihre Bedeutung haben."*
>
> [Formal axiomatics needs with necessity the contentual axiomatics as its supplement, because only through this is the instruction for the selection of formalisms given and furthermore, for an existing formal theory, only the application to a field of reality. On the other hand, we cannot stop at the contentual axiomatics, because in science we are dealing, if not steadily, then at least predominantly with those theories which do not completely reflect the real facts of the case, but which represent a simplifying idealization of the facts and have their meaning in them.]

Already, a look at Euclidean geometry shows us that this is a theory *"die inhaltlich und anschaulich gemeint ist"* [*'which is meant contentually and perceptively'*] (HILBERT-BERNAYS, I, op. cit., p. 20), even if it deals with *idealizations* that no longer can be perceived with the senses. However, the extent of idealization (for example, the breadthslessness and the infinite divisibility of lines, etc.) can only be made precise and accessible to a mathematical treatment by specifying formalisms.

What had been achieved was convincing, but, still, not all mathematicians accepted it.

HERMANN WEYL (1885-1955) would have liked to have *'stopped'* at a mathematics that is contentually conceived. In his *'Diskussionsbemerkungen'* (op. cit.) to a lecture by HILBERT in 1928, he looked back somewhat wistfully at the mathematics of past times and said that it had previously been *"ein System inhaltlicher, sinnerfüllter, einsichtiger Wahrheiten"* [*'a system of contentual, meaningful, insightful truths'*], and that, today, *"die Grenzen des inhaltlichen Denkens überall weit überschritten seien"* [*'the limits of contentual thinking have been far exceeded everywhere'*]. These limits had indeed been *'far exceeded everywhere'* in the meantime, but very much to the advantage of mathematics, because it had now become possible to realize the ideal of a really *'strict proof'*. FELIX KLEIN (op. cit., p. 51) wrote about this:

> *"Der Begriff der »Strenge« in unserer Wissenschaft und die Forderung des in ihm enthaltenen Ideals stammt von den Griechen. Sie verstanden darunter die rein logische*

Ableitung der ganzen Mathematik aus einer möglichst beschränkten Anzahl an die Spitze gestellter Voraussetzungen."

[The concept of *'rigour'* in our science and the request for the ideal contained in it originated with the Greeks. They understood it to mean the purely logical derivation of all mathematics from a limited number of preconditions placed at the top.]

The *'purely logical derivation*' of which KLEIN speaks is a chain of argumentations, all of which are already valid *because of their form.* Such a 'pure logic' was developed in the late 19th century and early 20th century by BOOLE, FREGE, GENTZEN, GÖDEL, HILBERT, PEIRCE, RUSSELL, SCHRÖDER, SKOLEM, TARSKI and many others, and is today called *Formal Logic* or *Symbolic Logic* or *Mathematical Logic*.

ALBERT EINSTEIN wrote even more decisively, in his famous lecture *'Geometrie und Erfahrung'* (1921, op. cit.):

"Der von der Axiomatik erzielte Fortschritt besteht nämlich darin, daß durch sie das Logisch-Formale vom sachlichen bzw. anschaulichen Gehalt sauber getrennt wurde; nur das Logisch-Formale bildet gemäß der Axiomatik den Gegenstand der Mathematik, nicht aber der mit dem Logisch-Formalen verknüpfte anschauliche oder sonstige Inhalt.

[The progress achieved by axiomatic theory consists in the fact that it has neatly separated that which is logical-formal from the factual or descriptive content; only that which is logical-formal constitutes the object of mathematics according to axiomatic theory, but not the descriptive or other content connected with the logical-formal.]

From this statement of EINSTEIN's, it becomes clear that *'the progress achieved by axiomatics'* in this case actually comes from an influence of logic on mathematics.

But the roots of contentual thinking are by no means ignored in the axiomatic foundation of a mathematical theory; only the formalisable aspects are clearly separated from the content-related specifications. The actual mathematical procedure has become transparent, controllable and verifiable in its possibilities and limits.

Apart from WEYL, other prominent mathematicians also expressed their disapproval. Some even vehemently opposed the formal standpoint, such as HEINRICH BRANDT (1886-1954), LUITZEN EGBERTUS JAN BROUWER (1881-1966), GOTTLOB FREGE (1848-1925), CARL LUDWIG SIEGEL (1896-1981) and a few others. In the next section we want to deal with FREGE's polemical attacks.

18.5 FREGE's polemics against HANKEL's formal standpoint

One of the most prominent opponents of the formal position was FREGE. For him, the *empty signs* of the formalists were *'inanimate, lifeless signs*' (*"entseelte Zeichen",* cf. FREGE *'Kleine Schriften'*, p. 311). He repeatedly made mocking and polemical comments about formalism. His polemics were mainly directed against HERMANN HANKEL (1839-1873), EDUARD HEINE (1821-1881) and CARL JOHANNES THOMAE (1840-1921).

HANKEL, e.g., wrote, in his *'Theorie der komplexen Zahlsysteme',* op. cit., part 1, 1867, p. 5), concerning the equation $x + b = c$, where b and c indicate concrete natural numbers:

"Es liegt auf der Hand, dass es, wenn $b > c$ ist, keine Zahl x in der Reihe 1,2,3,... giebt, welche die betreffende Aufgabe löst; die Subtraction ist dann unmöglich. Nichts hindert

uns jedoch, dass wir in diesem Falle die Differenz c – b als ein Zeichen ansehen, welches die Aufgabe löst, und mit welchem genau so zu operieren ist, als wenn es eine numerische Zahl aus der Reihe 1,2,3,... wäre."

[It is obvious that if $b > c$, there is no number x in the series *1,2,3,...* which solves the problem in question: subtraction is then impossible. Nothing prevents us, however, from seeing the difference $c – b$ as a sign which solves the problem and with which we can operate exactly as if it were a numerical number from the series *1,2,3,...* .]

FREGE objected that it was not the (empty) sign $c – b$, but only its 'content' that could solve the problem $x + b = c$. The *'content'* of a sign is, in FREGE's view, the totality of all *attributes* (i.e., essential properties) that the object designated by the sign possesses. For example, if there is no object that is designated by the sign *2 – 3*, then *2 – 3* is not the name of a number and, according to FREGE, it is

"... nur Tinte oder Druckerschwärze auf Papier, [und] *hat als solche physikalische Eigenschaften, aber nicht die, um 3 vermehrt 2 zu ergeben."*

[... only ink or printer's ink on paper, [and] has physical properties as such, but not those to give 3 when increased by 2.]

FREGE: *'Grundlagen der Arithmetik'*,Breslau 1884, op. cit., p. 107.

FREGE first required of the foundation of arithmetic that all of its objects must be presented, objects that are *as such* (*per se*) either positive numbers or negative numbers. In a second step, the essential properties of these objects have to be determined and all that is true among them has to be examined thoroughly. In a third step, the objects have to be defined and all that is true among these objects has to be written down as the true axioms of arithmetic. According to FREGE, axioms have to be true statements. [4] This is the procedure that has to be taken into consideration when a mathematical theory is to be founded according to the contentual point of view.

FREGE rejected an axiomatic foundation from a formal point of view, since such a foundation would only refer to the handling of linguistic signs and series of signs. In a model of such an axiom system, the objects would not necessarily have the desired properties *per se* (καθ' αὐτό), but only *accidentally* (*per accidens*), i.e., insignificantly (κατὰ συμβεβηκός, in the Aristotelian terminology) . [5]

FREGE himself tried to present the objects of arithmetic explicitly as logical entities, and to use them in the foundation of arithmetic from a *contentual point of view* (cf. Chapter 14). He believed that his approach was the only correct one and superior to HANKEL's

[4] In his letter of Dec. 27, 1899, FREGE wrote to HILBERT: *"Axiome nenne ich Sätze, die wahr sind, die aber nicht bewiesen werden, weil ihre Erkenntnis aus einer von der logischen ganz verschiedenen Erkenntnisquelle fliesst, die man Raumanschauung nennen kann."* See: FREGE: *'Wissenschaftlicher Briefwechsel',* 1976, op. cit., p. 63.

[5] The meaning of the term *'per se'* ('by itself') is best explained by a comparison with the introduction of money. A piece of money that is worth 20 £ (or 20 $ or 20 DM) could be a coin made out of gold and is, just as an object, worth what is printed on it. But it could be also just a piece of paper upon which it is printed that, in the game of exchanging goods, it plays the role of being worth 20 £. However, according to J.W.v.GOETHE, *Faust II,* 6037-6183, it was Mephisto who proposed the introduction of paper money.

formalized axiomatic approach. But FREGE was mistaken and was unable to achieve his goal, as we have seen in Chapter 14.

In contrast to FREGE, HANKEL had taken the *formal standpoint*. He also aimed at a satisfactory introduction of positive and negative numbers. His leading idea, however, was to purify the foundation of arithmetic by eliminating all extramathematical interferences *('außermathematische Einmischungen')*, in particular, those interferences that are caused by the use of colloquial language and its semantics. Such an interference is the understanding of the operation of *'subtraction'* in the colloquial sense of *'taking away',* known from daily life. HANKEL's idea was to replace that operation with another operation that is the same but can be introduced into mathematics on a formal and indisputably safe basis. Thus, he proposed to replace the contentual operation of *'taking away'* with an operation that, from a mathematical point of view, is the exact *converse* of 'addition'. In order to emphasize the distinction, he introduced the converse operation as a *lytic operation*[6], as he called it, a kind of *'Umkehrfunktion'*. Thus, the binary operation $x-y$ is not defined with reference to an extramathematical content as the result of taking the quantity y away from the quantity x, where all terms are understood in the usual sense of colloquial language, but simply by the formal stipulation: $(x-y)+y=x$, or, in other words, by putting:

$$x-y=z \quad \text{if and only if} \quad z+y=x\,.$$

In this way, HANKEL introduced the subtraction sign as an *empty sign* and not as a *symbol,* as it was treated in earlier times. HANKEL also introduced these new signs:

$$0-0,\ 0-1,\ 0-2,\ldots,\ 1-0,\ 1-1,\ 1-2,\ 1-3,\ldots,\ 2-0,\ 2-1,\ 2-2,\ 2-3,\ \ldots\ \text{etc.},$$

which he called *'purely mental numbers'* or *'purely formal numbers'*.[7] These numbers, together with the natural numbers, which are terms of the form ((...((1+1)+1)...+1)+1) with the usual abbreviations $2=1+1$, $3=2+1$, etc., form the totality of all whole numbers. Addition and multiplication is defined recursively as proposed by H. GRASSMANN in the year 1860. In this way, the arithmetic of whole numbers is introduced from a *nominalistic point of view.*

This means that only a denotation system is set up. It may be used in daily life, and thus denote concrete objects. That what HANKEL could not achieve was to present *'the'* positive and negative whole numbers, i.e., *'the'* unique realm of abstract or idealized numbers that are only mentally perceivable - if such a realm of objects exists at all. Notice that set theory

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[6] In Greek, 'lysis' (λύσις) means 'solution', 'loosening'. '

[7] The positive numbers, which belong to the *'science of quantities'* (*'Größenlehre'*), are called *'actuelle Zahlen'* (i.e., 'actual numbers') - cf. HANKEL *'Theorie der complexen Zahlsysteme'*, op. cit., p. 7. With the distinction *'actual - formal'*, he wanted to avoid the unfortunate terminology of *'positive & negative'* numbers. HANKEL also formulated a *principle of permanence*, in which it is stated that the purely formal numbers should obey the same arithmetical laws as the actual numbers (cf. HANKEL, loc. cit., p. 11, p.15), an idea that, in a slightly narrower sense, was first stated by GEORGE PEACOCK in 1830/1834. PEACOCK possibly formulated it under the influence of his friend CHARLES BABBAGE (1821) and his work on mechanical calculating machines, cf. J.M. DUBBEY, op. cit. Notice that the operations of a calculating machine obviously are not based on contentual definitions, but only on formal rules.

was not yet developed at that time and could not serve as a foundation of a formal semantics for his calculus of whole numbers.

FREGE criticised HANKEL for having presented only a system of denotations, but not a system of objects that are denoted by these denotations. Thus, FREGE criticised HANKEL for not having exhibited a system of objects that are *per se* whole numbers.

However, HANKEL gave a formal calculus in which the meaning is attached to the sign $c-b$ that solves the problem $x + b = c$. Thus, it is *not that the sign itself* (per se) contains the information that it solves the problem $x + b = c$, but rather the calculus witnesses that the sign *plays the role* of an object that solves the problem. Thus, HANKEL's contribution to the foundation of number systems may be understood as a precursor to *'structuralism'* (cf. Chapter 19). FREGE*'s* criticism is obviously unfounded here.

In the history of mathematics, since antiquity, it has become clear over and over again that, in founding a mathematical theory, one cannot start with the specification of the realm of objects to be investigated. The first thing one can do in the founding of a mathematical theory is to set up the *'framework of the relevant concepts'*, that is to say, by formulating the axioms of the theory with respect to its fundamental concepts. In a next step one can try to find interpretations or models of the theory, and this, for example, on the basis of set theory. Here, 'set theory' is understood as a formal theory that provides mathematics with a formal semantics.

18.6 Résumé

We want to return to the question posed at the beginning of this chapter, *whether, when founding a mathematical theory, the formalisms serve to* ***reproduce*** *contents given in advance as precisely and clearly as possible or, vice versa, whether they serve to* ***introduce*** *the contents*.

Our explanations have shown that these are not standpoints that exclude each other, but rather standpoints that should be supplementary one to the other!

In the combination of these points of view, the contribution that comes from the contentual point of view belongs to the *prolegomena*, that is, to that which precedes the actual activity of a mathematician.

The actual activity begins only when the basis of the theory envisaged has been fully elaborated. This actual activity consists in solving problems on the basis provided and, if necessary, checking whether the problems presented can be solved at all on this basis. To achieve this goal, the theories must have a high degree of formal accomplishment. In this regard, DAVID HILBERT has provided decisive motivations and contributions, which we will discuss in Chapter 20.

Some mathematical theories, however, are developed without being based on extra-mathematical experiences. An example is the theory of complex numbers $x + iy$ (x,y denote real numbers, $i = \sqrt{-1}$). BOMBELLI had constructed this theory from a formal point of view by specifying the rules of calculation (see above, section 18.1).

Many other mathematical theories are based on predetermined contents that had to be modified, however, in order to make them accessible to mathematical thinking. Examples

of these are HILBERT's geometry and ZERMELO-SKOLEM-FRAENKEL's set theory ZSF.

The geometry of the Egyptians, Babylonians and possibly also of other early cultures in pre-Pythagorean time was based on sense-perceptible constructions with a compass, a ruler and a right angle. The objects of such a geometry were sense-perceptible figurations. This is a geometry performed from a contentual point of view and may be counted among the prolegomena of a mathematically-conceived geometry.

EUCLID's geometry aimed at a mathematical theory based on concepts. It is still a theory that is meant contentually. A presentation of geometry from a formal point of view was achieved only at the end of the 19th century, in particular, by D. HILBERT (cf. Chapter 20).

Naive set theory, as worked out by BOLZANO, DEDEKIND and others, is based on the content of the concept of a *'set'* in its colloquial meaning. The content of that concept, however, leads to contradictions, and hence cannot be taken as the basis of a mathematical theory of sets. The naive concept of a 'set' was considerably modified by CANTOR, who shaped it in a new and different way on the basis of his theological-philosophical thinking (cf. Chapter 15), unfortunately, with reference to extra-mathematical contents. Thus, CANTOR's set theory is still a theory founded from a contentual point of view. It may be counted among the prolegomena of a more mathematically-conceived set theory, such as ZSF. The formal ZERMELO-SKOLEM-FRAENKEL set theory ZSF is *not designed to reproduce the contents of the naive concept of a 'set',* as used by BOLZANO and DEDEKIND*, but to reproduce* CANTOR's concept of a 'set'. This is done by eliminating the extra-mathematical contents from theology, and introducing the concept of a 'set' by means of axioms in an implicite manner.

Finally, we come back to the two citations that we placed at the beginning of this chapter. The statement by COUTURAT is a harsh response to BERKELEY's statement, since it cannot be denied that even a mathematical discipline that is drafted from a formal point of view is concerned with ideas and contents, however suitably idealized and formalized. In fact, *what cannot be formalised cannot be dealt with by means of mathematics*.

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Chapter 19

DEDEKIND and the emergence of Structuralism

"Als letzter abstrakter Ausdruck bleibt in jeder Kunst die Zahl."
[At the last abstract expression, in every art the number remains.]

WASSILY KANDINSKY*: 'Über das Geistige in der Kunst',* Piper Verlag München, 1912, p. 113.

RICHARD DEDEKIND (1831-1916) published a small booklet in 1888 in which he gave a sophisticated and novel answer to the fundamental question of what the (natural) numbers really are. The full title in German reads: *"Was sind und was sollen die Zahlen?"* The title is not easy to translate, because of its elliptical form. The following translation, however, seems to be appropriate: *'What are the numbers and what are they meant for?'*

The title is also a bit 'bold', because the (natural) numbers have been known - or seem to have been known - to humankind since time immemorial, and the title suggests that the question of what the numbers really *'are'* has still not been convincingly and finally answered.

Also DEDEKIND's answer to the first question posed in the title of his booklet was a bit 'bold', or at least 'surprising', in his time, because he was convinced that natural numbers exist only as the members of systems that belong to a certain *type of structured system* (or set) - cf. section 7 in DEDEKIND's letter from February 27, 1890, to HANS KEFERSTEIN. There does not exist a 'canonical' prototype, although one can choose one according to one's liking, as is also usually done.

In addition, DEDEKIND was also convinced that it is not possible to define the natural numbers individually and in isolation from each other, but that it is only possible to characterize them as a system (or set), namely, as a *'structured set'* (*"von aufeinander bezogenen Dingen"*[1], i.e., of things *'related to each other'*).

In doing so, he became the founder of the position that, in the early 20th century, was

[1] DEDEKIND expressed it in the preface of his booklet, p. VIII, with the same words that ARNOLD SCHÖNBERG used them much later, ca. 1923, in the case of his twelve-tone music.

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U. Felgner, *Philosophy of Mathematics in Antiquity and in Modern Times*,
Science Networks. Historical Studies 62, https://doi.org/10.1007/978-3-031-27304-9_19

called *'structuralism'*.[2] This structuralism very soon became the dominant standpoint in algebra, geometry, topology and, a little later, in almost all other mathematical disciplines - and has remained so ever since.

RICHARD DEDEKIND was born in Braunschweig (Brunswick, Germany) on October 6, 1831. He studied in Göttingen between 1850 and 1852, and wrote his doctoral thesis under KARL FRIEDRICH GAUSS. He took his Ph.D. degree in Göttingen in 1852. In 1854, he was granted the Habilitation by Göttingen University. He taught there as a 'Privatdozent', in the subsequent years 1854-1858, his first courses on Probability-theory, Geometry and Algebra (including Galois-Theory and Group-Theory). In 1858, he received the offer of a 'Dozentur' at the Zürich Polytechnikum (ETH) and, in 1862, was promoted to full professor. Despite the distinguished reputation of this chair, he followed a call in 1862 to a professorship at the Polytechnikum in Braunschweig. He remained there, although he received invitations for professorships at the Universities in Hannover, Straßburg im Elsaß, Gießen, Karlsruhe and Göttingen (twice). DEDEKIND retired in 1894. He died in Braunschweig at the age of 84 years on February 12, 1916. Aspects of his life and work are reported in the volume *'Richard Dedekind, 1831/1981'*, op. cit. published by W. SCHARLAU. The publication of his *'Collected Mathematical Works'*, edited by ROBERT FRICKE, EMMY NOETHER and OYSTEIN ORE, in 3 volumes, was accomplished in 1930, 1931 and 1932.

Let us now begin with a presentation of DEDEKIND's contribution to the number concept and the conception of structuralism.

19.1 The traditional concepts of the number

People have known how to count since primeval times and have used numerical words for counting. They have also used numerals and ciphers for calculating for a very long time. But the *concept of the number* had not yet been obtained thereby.

What are numbers? Where do they exist, and in what sense do they exist? Are the linguistic expressions *'one', 'two', 'three',....* themselves the numbers? - In other languages, other expressions are used, such as *'uno', 'dos', 'tres', ...* or *'uno', 'due', 'tre',* ... etc., and this suggests that the linguistic expressions are perhaps just the different *names* of abstract or idealized things that are called 'numbers'.

THALES, PLATO and EUCLID defined numbers as (finite) sets of units. But, then, the question arises, what are 'units' and what is their essence and nature? Are the things of the real world the 'units' that should always be regarded as *'one'* when counted, as many textbooks of earlier centuries say? Then, numbers would be nothing more than the finite collections of objects of the real world, i.e., the 'numbers' with which - as PLATO wrote - merchants deal in their daily life (cf. Chapter 2). Or, as JACQUES OZANAM (1640-1717) put it, in his *'Dictionnaire Mathématique'* (Amsterdam, 1691, p. 21): *"Le nombre est l'assemblage de plusieurs choses de même genre"* (*'Number is the collection of several*

[2] DEDEKIND may be counted among the *structuralists.* In his work, however, one also finds echoes of *logicism, psychologism* and *formalism.* These directions, however, cannot be attributed to him without restriction, cf. Chapter 14.

things of the same kind'). But are numbers really such objects?

Perhaps numbers are the (finite) sets of abstract, indistinguishable units that are there only for our thinking, as the mathematicians and philosophers of antiquity taught (cf. Chapters 2, 3 and 15)?

Or is the number *'three',* for example, not rather the *idea of a trinity,* i.e., the *idea* of a collection of three different objects, as LEONHARD EULER (following JOHN LOCKE) wrote in his *'Letters to a German Princess'* (there in the 100th letter), i.e., that which is common to all three-element sets?

In the enlightened Europe of the 18th and 19th centuries, this view very soon prevailed. It penetrated the mathematical literature, and even schoolbooks. In all these works, numbers were introduced as the *ideas* that the human soul extracts from all totalities of equipotent (finite) collections, by emphasizing only the fact that its elements are pairwise different and *abstracting* from all other characteristics.

But, what are *ideas*? - and is something like *'the idea of a trinity',* for example, a real mathematical object at all?

19.2 DEDEKIND's simply infinite systems

To RICHARD DEDEKIND, we owe a fundamentally new idea for introducing the concept of the number. After he had published an introduction to the concept of real numbers in 1872, entitled *'Stetigkeit und irrationale Zahlen'*, he published his new epoch-making booklet in 1888, to explain the concept of natural numbers: *'Was sind und was sollen die Zahlen?'* He had already written a first draft (15 pages) in the years 1872-1878.

When he saw the treatises *'Zählen und Messen'* by HERMANN VON HELMHOLTZ (1821-1894) and *'Über den Zahlbegriff'* by LEOPOLD KRONECKER (1823-1891), both published in 1887, he compared them with his own original outline. He was not convinced by the nominalist views of the two authors' concept of the number. DEDEKIND immediately began - as late as June/July 1887 - to revise his own old sketches. He wrote the third and final version in August/October 1887. The booklet appeared in print in January 1888.

In the preface, he gave, as his main answer to the questions asked in the title (an answer that he had already given in the first draft from 1872, see P. DUGAC, op. cit. 1976):

> *"Die Zahlen **sind** Schöpfungen des menschlichen Geistes; sie **sollen** als ein Hilfsmittel dienen, die Verschiedenheit der Dinge leichter aufzufassen."* (emphasis added)
>
> [Numbers **are** creations of the human spirit; they are **meant** to serve as a tool to make the diversity of things easier to grasp.]

Following CARL FRIEDRICH GAUSS (1777-1855), who wrote, in a letter from April 9, 1830, to FRIEDRICH WILHELM BESSEL (1784-1846), that *'the number ... is the product of our mind'* (*"dass die Zahl ... unsers Geistes Product sei"*), DEDEKIND also considered numbers a *'creation of the human mind'* (eine *"Schöpfung des menschlichen Geistes"*) - cf. KRONECKER, op. cit., p.253, and DEDEKIND *'Was sind und was soll die Zahlen?',* #73. [3]

According to DEDEKIND, the *'creation'* of numbers does not take place by creating them

[3] With #1,..., #172, we designate the individual sections of DEDEKIND's booklet *'Was sind und was sollen die Zahlen?'* in the numbering given by DEDEKIND himself.

individually, e.g., through abstraction, or whatever. The *'creation'* of numbers takes place through the formation of the *'abstract type'* of a certain structured set. The structured set itself is a so-called *'simply infinite system'* (*"einfach-unendliches System"*), which will be understood - or interpreted - as a collection of *all* natural numbers. The iridescent concept of an *'abstract type'* must, of course, be explained and clearly defined.

What is constitutive for the concept of the number (for DEDEKIND, in contrast to all of his predecessors, including GRASSMANN, HANKEL, HELMHOLZ, KRONECKER and others, cf. Chapter 18) is the fact that numbers are connected by the *principle of mathematical induction* (for more on 'induction' see U. FELGNER, 2012, op. cit.). This principle makes use only of the concatenation of numbers by the *successor operation*, i.e., the transition from a number n to its successor $n + 1$. The totality of all numbers is thus to be introduced in such a way that it is equipped with the 1-ary (i.e., 1-placed) operation of transition from a number to its successor and that, in this totality, the principle of mathematical induction is valid! DEDEKIND had the brilliant idea that the validity of the *principle of mathematical induction* in the totality of all numbers is equivalent to the *'simplicity'* of that totality with respect to the (algebraic) 1-ary operation of passing from an element to its successor. Here, in fact, *'simplicity'* is understood in the usual algebraic sense, and means the inexistence of proper substructures. (For more details, see below.)

The totality of all numbers does not have to be equipped with the operations of addition and multiplication. These two binary (i.e., 2-placed) operations can easily be obtained (according to H. G. GRASSMANN, 1860) through recursion as follows (for any fixed number n):

$$n + 0 = n \quad \& \quad \text{for all } x\text{:} \quad n + (x + 1) = (n + x)+1,$$
$$n \cdot 0 = 0 \quad \& \quad \text{for all } x\text{:} \quad n\cdot(x+1) = n \cdot x + n.$$

When things are counted, they are *chained* in a certain way, the first element being followed by the second, the second by the third, etc. While *numbers* (or, at least, number-words) are needed for counting, the more elementary concept of *concatenation* can be described without the use of numbers, namely, by a one-to-one mapping φ. Starting from the first thing a, by applying φ, one reaches the next thing $\varphi(a)$, and then, by applying φ again, a further thing $\varphi(\varphi(a))$, etc. Hence, the concatenation allows us to imitate the operation of transition from a number n to its successor $n + 1$, i.e., allows us to imitate the process of counting. This led DEDEKIND to the following fundamental concepts.

Definitions. (1) A mapping φ from a set A into a set B is called *one-to-one* (or, in DEDEKIND's words, #26: *"ähnlich"* (*'similar'*) or *"deutlich"* (*'distinct'*)), if exactly one element of B is always assigned to each element of A as its image, and different elements of A always have different images in B.

(2) A set M is called *infinite* (we will say 'DEDEKIND-*infinite*', in order to distinguish this definition from other definitions of 'infinity') if there is a one-to-one mapping of M onto a proper subset A of M, i.e., $A \subseteq M$ and $A \neq M$, (cf. DEDEKIND, #64).

According to (2), a set M is DEDEKIND-*finite* if each one-to-one mapping from M into itself is necessarily surjective, or, in other words, if each one-to-one mapping from M into

itself maps *M onto* itself. It is remarkable that, in (2), the concepts of *finiteness* and of *infinity* are defined without reference to natural numbers!

We add the following definition, which is not in DEDEKIND's booklet. Notice that these DEDEKIND-triples are not considered as *'structures'* in the modern sense. (However, equipped with a suitable formal language, they could be turned into a structure.)

(3) A triple (M, φ, a) is called a DEDEKIND-*triple* if $a \in M$ and φ is a one-to-one mapping of the set M onto a proper subset of M such that $a \notin \{\varphi(x); x \in M\}$.

The following definition is credited to DEDEKIND and is of fundamental importance.

(4) A DEDEKIND-triple (M, φ, a) is called *simply infinite* if M is the only subset of M that contains the displayed element a and is closed under φ (cf. DEDEKIND, #71).[4]

19.3 Properties of simply infinite systems

In the following theorems #71, #72, #80, #132 and #133, some of the essential properties of simply infinite systems are collected.

Theorem (DEDEKIND, #71) *Let M be a non-empty set, φ be a mapping defined on M and a be an element of M. Then,* (M, φ, a) *is a simply infinite* DEDEKIND-*triple if and only if the following four properties are satisfied:*

- (α) *If n is an element of M, then φ(n) is also an element of M;*
- (β) *Each subset X of M, which contains a and, with each x, also its image φ(x), is identical with the whole set M;*
- (γ) *The displayed element a is not an image under φ;*
- (δ) *The mapping φ is one-to-one.*

Theorem (DEDEKIND, #72) *Every* DEDEKIND-*infinite set has simply infinite subsets.*

Proof. Let M be an arbitrary DEDEKIND-*infinite* set and φ be a one-to-one mapping from M onto a proper subset of M. Select an arbitrary element a such that $a \in M$ and also $a \notin \{\varphi(x); x \in M\}$, and let D_a be the intersection of all subsets K that contain a and are closed under φ,

$$D_a = \bigcap \{K; a \in K \ \& \ \{\varphi(x); x \in K\} \subseteq K \subseteq M\}.$$

Then, (D_a, ψ, a), where ψ is the restriction of φ to D_a, is a *simply infinite* subset of M, Q.E.D.

Thus, if there are DEDEKIND-infinite sets, then there are also simply infinite DEDEKIND-

[4] In his first draft of 1872/1878, DEDEKIND still called subsets of M, which are *closed under φ,* 'groups' in free reference to ÉVARISTE GALOIS (1831). He borrowed the term *'simplicity'* just as freely from CAMILLE JORDAN (*'Traité des Substitutions et des Équations algébriques',* Paris 1870, p. 41). In order to avoid misunderstandings with group theory, which was only just emerging at that time, DEDEKIND changed the terminology - as stated above. Notice that *'simplicity'* is still used here in the group-theoretical sense.

triples. The question, however, as to whether DEDEKIND-infinite sets exist at all cannot be easily answered. We will discuss this question in section 19.5.

Theorem #72 shows that the chain D_a can be represented 'from above' as an intersection, but not 'from below' in the form $D_a = \{a, \varphi(a), \varphi(\varphi(a)), \varphi(\varphi(\varphi(a))), ...\}$, since this would require a definition by recursion, which is not yet available.

Corollary (DEDEKIND, #80): *Let* (M, φ, a) *be a* DEDEKIND-*triple.*
(i) *If, in* (M, φ, a), *the* ***P****rinciple of* ***M****athematical* ***I****nduction* (PMI) *holds in the following form:*

$$\text{(PMI)} \qquad \forall X \subseteq M \left((a \in X \;\; \& \;\; \forall x \, (x \in X \Rightarrow \varphi(x) \in X) \right) \Rightarrow X = M),$$

then M is simply infinite.
(ii) *Conversely, if* (M, φ, a) *is simply infinite, then it satisfies the principle of mathematical induction* (PMI).

Proof. (i): In (PMI), one only has to choose $X = D_a$. (ii) is obvious, Q.E.D.

The next aim is to show that all simply infinite DEDEKIND-triples resemble each other very strongly, and, in fact, form a single *type* when treated from an abstractionist point of view. The following theorem is a first indication of this resemblance. This theorem may be considered as a side-piece to GEORG CANTOR's early discoveries from 1873/1874 concerning the cardinalities of the sets of integers and of real numbers.

Theorem (DEDEKIND, #132): *If* (A, φ, a) *and* (B, ψ, b) *are simply infinite* DEDEKIND-*triples, then there exists a one-to-one mapping from A onto B*.

Proof. The mapping $f: A \rightarrow B$, which is defined by recursion (#126) as follows:

$$f(a)=b \quad \& \quad \text{for all } x \in A: \;\; f(\varphi(x)) = \psi(f(x)),$$

is the bijective mapping we are looking for, because the displayed elements (also called *'base elements'*) are mapped onto each other, $f(a)=b$, and, through mathematical induction, which is valid in (A, φ, a), as well as in (B, ψ, b), as DEDEKIND shows in #59, #60 and #80, it follows that f is surjective, Q.E.D.

The converse of the above theorem is also true:

Theorem (DEDEKIND, #133): If (A, φ, a) *is any simply infinite* DEDEKIND-*triple and M is an arbitrary set, and if there exists an arbitrary one-to-one mapping f from A onto M, then it is possible to define, on M, a one-to-one mapping* θ *from M into M and to select an element m from M such that* (M, θ, m) *is also a simply infinite* DEDEKIND-*triple.*

Proof. In fact, define θ to be $\theta = f \circ \psi \circ f^{-1}$ and $m = f(a)$, Q.E.D.

According to DEDEKIND, both theorems, #132 & #133, imply that the totality of all simply infinite DEDEKIND-triples forms a *'class'* (*"eine Klasse"*). The concept of a *'class'* is

defined in #34, and is meant here, in the sense of CANTOR (cf. 'Ges. Abhandl.', p. 141), as a *'numberclass',* i.e., an equivalence-class of all sets that are 'equivalent' in the sense of equipotency (that is: having the same cardinality). Thus, the totality of all sets *M* that can be expanded to a DEDEKIND-triple (*M, φ, m*) is an equivalence-class of sets with respect to the equivalence-relation of equipotency. This means that this totality of all sets *M* that can be expanded to a DEDEKIND-triple (*M, φ, m*) forms a *type of sets with respect to equipotency.*

This statement is strengthened in the next theorem #134, where it is proved that all simply infinite DEDEKIND-triples form a *unique type with respect to their abstract algebraic properties*. This is done by showing that the one-to-one mapping *f* defined in the proof of theorem #132 not only 'respects' the displayed basic elements *a* and *b* and the fundamental successor functions *φ* and *ψ,* but also 'respects' *all* further relevant 'contents' (*"Inhalte" und "Bedeutungen"*). We, today, would say that *f* is not only a one-to-one mapping, but also an 'isomorphism' between these triples, assuming that these triples may be considered as 'structures' in the modern technical sense. However, the triples (*M, φ, a*) and (*N, ψ, b*) are not considered as 'structures'. Accordingly, DEDEKIND does not speak of 'isomorphisms'. (Notice that 'structures' are defined with respect to *formal languages* containing *'empty signs',* which are interpretable in these structures. This indicates that, in the formation of a structure, a good portion of abstraction is already built in, which is not the case with DEDEKIND-triples.)

It must have been rather difficult for DEDEKIND to express his aims sufficiently clearly, since, at his time, the relevant terminology was not yet available - but he paved the way to its introduction. What is needed is an explicit or implicit reference (or allusion) to some kind of abstraction, because the above defined mapping *f* can only transport those contents that are expressible in a language that contains names for the successor functions and the base elements as the only non-logical signs. Thus, what is needed is *abstraction from those contents* that are irrelevant in the present context.

19.4 The different concepts of *'abstraction'*

DEDEKIND's use of the term *'abstraction'* is explained by him in the following definition:

> **Erklärung** (DEDEKIND, #73): *"Wenn man bei der Betrachtung eines einfach unendlichen, durch eine Abbildung φ geordneten Systems N von der besonderen Beschaffenheit der Elemente gänzlich absieht, lediglich ihre Unterscheidbarkeit festhält und nur die Beziehungen auffaßt, in die sie durch die ordnende Abbildung φ zueinander gesetzt sind, so heißen diese Elemente* ***natürliche Zahlen****.* (....)*".* (emphasis added)
>
> [**Definition**: 'If, in the consideration of a simply infinite system *N* ordered by a mapping *φ*, one entirely disregards the special character of the elements (of *N*), retaining only their distinguishability and considering only the relationships in which they are placed to one another by the ordering mapping *φ*, then these elements are called ***natural numbers*****'** (.....)]

This definition[5] needs some further comments in order to be fully understandable in its

[5] Notice that DEDEKIND's formulation here is rather close to the formulation of a related definition given by RUDOLF LIPSCHITZ (1832-1903) on the first page of his *'Lehrbuch der Analysis'*, Bonn 1880, and also to the formulation of the concept of the cardinal-number as given by GEORG CANTOR in 1895 (see below). This shows how widespread this understanding of the process of *'abstraction'* (in the sense of

meaning and its significance.

In the second half of the 19th century, the term *'abstraction'* appeared in mathematical papers and books with rather different meanings.

In algebra, e.g., one spoke of *'abstract groups'* in order to emphasize that the particular definition of 'group-multiplication', e.g., as a 'multiplication' of matrices or as successive executions of mappings in the case of permutation groups, etc., is not essential and, hence, will be completely disregarded. This is a kind of abstraction in which certain phenomena are withdrawn in thought. This is abstraction in the sense of ARISTOTLE, as *aphairesis* (ἀφαίρεσις, cf. Chapt. 3). For more details, we refer the reader, e.g., to HANS WUSSING: *'Die Genesis des abstrakten Gruppenbegriffs'*, op. cit. Notice that DEDEKIND was familiar with this kind of abstraction, which he had already used in his early lectures on GALOIS-theory (cf. WALTER PURKERT, 1976, op. cit.).

In set theory, e.g., GEORG CANTOR spoke of *"Abstraktionsakte"* ('acts of abstraction') when he introduced cardinal numbers (*"Mächtigkeiten"*), as well as ordinal numbers (cf. CANTOR's *'Gesammelte Abhandlungen'* (Berlin 1932, p. 141, pp. 282-283, p. 320, pp. 379-381, pp. 441-442). For CANTOR, these are *acts of abstraction* in which *'one completely disregards the particular nature of the elements and the ordering in which they are given, retaining only their distinguishability'*. It is a kind of abstraction (related to abstraction à la JOHN LOCKE) in which a given object loses (!) certain characteristic properties, with the consequence that, thereby, the object is turned into a new object.

For CANTOR, the cardinal numbers are *"Allgemeinbegriffe, die sich auf Mengen beziehen"* ('general ideas referring to sets'), as he wrote in his letter from April 2, 1888, to GIULIO VIVANTI. For him, the cardinal numbers are, hence, created by an act of abstraction in his sense, as explained above. For DEDEKIND, however, the natural numbers are basically of algebraic origin: they are the members of a simply infinite DEDEKIND-triple, with the proviso that, when looking at them, all of their properties are 'forgotten', with the sole exception of those properties in which the base element and the successor-operation are the only non-logical objects occurring. This means that, in DEDEKIND's definition #73, abstraction is meant in the sense of ARISTOTLE, as *aphairesis* (ἀφαίρεσις, cf. Chapt.3) without creating new objects.

In #73, we read that DEDEKIND chooses an arbitrary simply infinite DEDEKIND-triple ($\mathcal{N}, \varphi, 1$) and completely disregards the particular nature of its elements, retaining only their distinguishability and considering only the relationships in which they are placed to one another by the ordering mapping φ, defining this (arbitrarily chosen) *abstract triple* as **the** *prototype* for the sequence of natural numbers.

We repeat that, in the process of abstraction executed here, no new DEDEKIND-triple is created. But it is highly interesting to note that, in an earlier draft of his manuscript,

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JOHN LOCKE, cf. Chap. 9) actually was at that time. We reported on LIPSCHITZ's formulation in Chapter 13, section 13.5. Notice that this kind of abstraction was also used by LEONHARD EULER in his *'Letters to a German Princess'* (1768-1772), as well as by CHRISTIAN WOLFF in his Textbook *'Anfangsgründe aller Mathematischen Wissenschaften...'* (1710 etc.) and by ABRAHAM GOTTHELF KÄSTNER in his Textbook *'Anfangsgründe der Arithmetik, ...'* (Göttingen 1758, etc.), with explicit references to LOCKE. But, in DEDEKIND's formulation, a different kind of abstraction is meant: Not individual numbers are created by an act of abstraction but structured systems can be viewed as number-systems.

DEDEKIND referred to abstraction in the sense of CANTOR or of LOCKE. In fact, WILFRIED SIEG and REBECCA MORRIS discovered (2018, op. cit.) that, in a previous handwritten version, DEDEKIND wrote:

> *"Da durch diese Abstraktion die ursprünglich vorliegenden Elemente n von N (und folglich auch N selbst in ein neues abstraktes System* $\mathfrak{N}$*) in neue Elemente* $\mathfrak{n}$*, nämlich Zahlen umgewandelt sind, so kann man mit Recht sagen, daß die Zahlen ihr Dasein einem freien Schöpfungsacte des Geistes verdanken."*
>
> [Since by this abstraction the originally given elements *n* of *N* (and hence *N* itself is turned into a new abstract system $\mathfrak{N}$) are turned into new elements $\mathfrak{n}$, namely into numbers, one is justified in saying that the numbers owe their existence to an act of free creation of the mind.]

It is noteworthy here that DEDEKIND spoke of *'new elements'* and a *'new system'* that is created through the process of abstraction. In that previous version, 'abstraction' meant subtracting certain properties from given objects. As soon as that kind of abstraction is carried out, the previously given objects no longer possess certain properties and, hence, are different from those that they had been before. This is abstraction in the sense of CANTOR (or of LOCKE). It was also the kind of abstraction DEDEKIND applied in some of his previous publications (e.g., in his booklet on *'Stetigkeit und irrationale Zahlen',* 1872).

But, when he did his ultimate revision of his manuscript in 1887, he replaced the above-mentioned passage with the one that can be presently found in #73. In that final version, there is no mention of a 'new' system that is created in the process of abstraction, and this indicates that DEDEKIND has decided to follow that understanding of the term 'abstraction' that he found among his colleges in algebra, namely, in the sense of ARISTOTLE.

The last step in answering the main question as to what numbers really *'are'* is taken in #134.

Theorem (DEDEKIND, #134): *Any theorem about numbers that is valid in the abstract simply infinite* DEDEKIND-*triple* ($\mathcal{N}$*,* φ*, 1*) *is also valid in any other simply infinite* DEDEKIND-*triple.*

Proof. Let Φ be a statement that is true in the abstract simply infinite DEDEKIND-triple ($\mathcal{N}$*,* φ*, 1*), which was chosen in #73. By the abstractness of the triple, we know that the only non-logical symbols occurring in Φ are the names for the displayed constant *'1'* and the name for the displayed successor-function *'*φ*'*. If (*M*, θ, *m*) is any other simply infinite DEDEKIND-triple, then, according to theorem #132, there is a one-to-one mapping from $\mathcal{N}$ onto *M* that maps *1* onto *m* and 'respects' both successor-functions. If, in Φ, we substitute any occurrence of the constant *1* with *m* and any occurrence of φ with θ, then, through mathematical induction (PMI), it follows that the statement, obtained through substitution in the described way, is true in (*M*, θ, *m*), Q.E.D.

This theorem has the consequence that, in all abstract simply infinite DEDEKIND-triples, the same theorems hold. All of these abstract simply infinite DEDEKIND-triples are, hence, *of the same type,* and, according to theorems #132 and #133, all of these abstract simply

infinite DEDEKIND-triples form a unique type. This is the *'abstract type'* of all simply infinite DEDEKIND-triples. It follows that any abstract simply infinite DEDEKIND-triple can function as a prototype, because they are all - as we would say today - isomorphic (they are all *"Systeme vom selben Typus"*).[6] Thereby, the answer to the first question, posed in the title of the booklet, is found: *'numbers'* are nothing more than members of arbitrary abstract simply infinite DEDEKIND-triples.

We also repeat that, for DEDEKIND, the *natural numbers* are never definable individually, but always only as members of a system of all numbers. Such a system of numbers is formed by the elements in a symply infinite triple in which the four DEDEKIND fundamental conditions (α), (β), (γ), (δ) from #71 are valid.

DEDEKIND answered the question of what the natural numbers *'are'* in a rather surprising and novel way: the natural numbers are not numbers *per se*, but only *objects that play the role* of being numbers in an abstract simply infinite DEDEKIND-triple. For example, an arbitrary object can only be named *'3'* if it belongs to an abstract DEDEKIND-triple in which the four DEDEKIND fundamental conditions (α), (β), (γ), (δ) from #71 are valid, and if it is the third element in the ordering given by the successor function φ.

19.5 The problem of the existence of infinite systems

But, one problem is left open so far, namely, whether DEDEKIND-triples exist at all. This is a rather critical question.

DEDEKIND had claimed, in #66, that such triples exist and referred to his own world of thoughts, *W,* as an example (leaning upon BOLZANO's *'Paradoxien des Unendlichen'*, 1851, §13). He meant that, with every thought, the thought about this thought also belongs to his world of thoughts, and thus a one-to-one 'successor function' is defined on *W*.

In his letter to HANS KEFERSTEIN from February 27, 1890, DEDEKIND commented on it (in the 7th section):

> *"Nachdem in meiner Analyse der wesentliche Charakter des einfach unendlichen Systems, deßen abstrakter Typus die Zahlenreihe $\mathcal{N}$ ist, erkannt war (#71, #73), fragte es sich: existiert überhaupt ein solches System in unserer Gedankenwelt? Ohne den logischen Existenz-Beweis würde es immer zweifelhaft bleiben, ob nicht der Begriff eines solchen Systems vielleicht innere Widersprüche enthält. Daher die Nothwendigkeit solcher Beweise (#66, #72 meiner Schrift),"*
>
> ['After in my analysis the essential character of the simply infinite system, whose abstract type is the number sequence $\mathcal{N}$, had been recognized (#71, #73), the question arose whether such a system exists at all in the realm of our thoughts? Without a logical proof of existence it would always remain doubtful whether the concept of such a system would perhaps contain internal contradictions. Such proofs are, hence, necessary (#66, #72 of my booklet)'.]

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[6] Compare, for example, EUGEN NETTO: *'Die Substitutionentheorie und ihre Anwendung auf die Algebra',* Leipzig 1882, p. 3 & p. 133, or ERNST STEINITZ: *'Algebraische Theorie der Körper',* Crelles Journal f.d.reine u. angew. Math., vol. 137 (1910), p. 167-309, Chapter I, § 1, or also WILLIAM BURNSIDE: '*Theory of Groups of finite order', Cambridge*1911, p. 21.

However, this totality *W*, if it were to be considered as a set, would give rise to an inconsistency (which was first pointed out by GEORG CANTOR in his letter from August 3, 1899, to DEDEKIND - cf. G. CANTOR *'Briefe',* op. cit., pp. 407-411, pp. 412-415. cf. also Chapter 15, section 15.4). ZERMELO (1917, op. cit.), in his exposition of DEDEKIND's theory of natural numbers, commented on it as follows:

> *"Die Existenz unendlicher Systeme, auf der seine Theorie der Zahlenreihe beruht, will Dedekind, anstatt sie einfach axiomatisch zu postulieren, auf das Beispiel unserer »Gedankenwelt«, d.h. die Gesamtheit alles Denkbaren, begründen. Er will die Existenz unendlicher Systeme beweisen, indem er jedem Element s seines Denkens den Gedanken zuordnet, daß s Gegenstand seines Denkens ist."*
>
> ['Instead of simply postulating by means of axioms the existence of infinite systems, on which his theory of the sequence of numbers rests, Dedekind wants to found it upon the example of our realm of thoughts, i.e. the totality of all thinkable. He wants to prove the existence of infinite systems by relating to each element *s* of his thinking the thought which has *s* as the object of his thinking.]

ZERMELO adds that such a realm of thoughts cannot be considered as a 'system' in the same sense as a 'set' is considered in Set Theory, since, otherwise, this would give rise to the appearance of antinomies such as the well-known *ZERMELO-RUSSELL-antinomy* (cf. Chapters 14 & 15). The appearance of such antinomies in DEDEKIND's theory of natural numbers had the effect that, for eight years, he stopped the re-edition of his booklet. However, he was willing to accept a third edition much later in 1911. There, in the preface to the 3rd edition, he confessed that *'his confidence into the internal harmony of our logic'* is not shattered by the appearance such antinomies, and that he believes that a careful investigation of the *'creative power of our human mind'* [7] (*"die Schöpferkraft des Geistes"*) will lead to a rehabilitation of his booklet. The desired rehabilitation took place, however, by applying a rather different remedy, and this was axiomatic set theory.

Simply infinite systems, which are sets, cannot be constructed on the basis of 'pure thinking' (of pure logic) alone, because, as DEDEKIND shows, they would be models of arithmetic and the consistency of arithmetic would follow. According to GÖDEL's incompleteness theorems (cf. Chapter 20), however, it is not possible to prove the consistency of arithmetic on the basis of pure logic. The construction of simply-infinite systems thus requires some principles of set-existence. In fact, on the basis of axiomatic set theory ZSF (see Chapter 15), simply infinite sets (which we allow to start here with the

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[7] The creative power of the human mind is emphasized by DEDEKIND in his well-known letter from January 24, 1888, to HEINRICH WEBER (op. cit., vol. 3, p. 489), where it is stated that *'we are of divine species and without any doubt possess creative power not merely in material things (trains, telegraphs) but quite specially in intellectual things. (...) We have the right to grant ourselves such power of creation.'* (The German text is: *"Wir sind göttlichen Geschlechtes und besitzen ohne jeden Zweifel schöpferische Kraft nicht blos in materiellen Dingen (Eisenbahnen, Telegraphen), sondern ganz besonders in geistigen Dingen.(...) Wir haben das Recht, uns eine solche Schöpferkraft zuzusprechen.(...)."*) According to DEDEKIND, in his booklet *'Stetigkeit und irrationale Zahlen'* (1872), the creative power of the human intellect comes to light in the creation of irrational numbers, and in his booklet *'Was sind und was sollen die Zahlen?'*, it comes to light in the creation of simply infinite triples.

number 0, however) can be specified very easily, for example,

$$\mathbb{N} \cong \{\emptyset,\{\emptyset\},\{\{\emptyset\}\},\{\{\{\emptyset\}\}\},\ldots.\},$$

with the construction rule $0 = \emptyset$ & $n + 1 = \{n\}$ (following ERNST ZERMELO, 1908), or

$$\mathbb{N} \cong \{\emptyset, \{\emptyset\}, \{\emptyset,\{\emptyset\}\}, \{\emptyset,\{\emptyset\},\{\emptyset,\{\emptyset\}\}\}, \ldots.\},$$

with the construction rule $0 = \emptyset$ & $n + 1 = n \cup \{n\}$ (following JOHANN VON NEUMANN, 1922, and also ZERMELO, who had already communicated this definition in a letter to PAUL BERNAYS in 1915).

These are, by now, well-known prototypes for the sequence of all natural numbers. In the first set, for example, $\{\{\{\emptyset\}\}\}$ plays the role of the number *3* and, in the second set, it is played by $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$. Thus, the nature of the respective objects does not matter; what matters are the positions in the infinite sequence.

19.6 The axiomatization of Arithmetic (DEDEKIND, PEANO)

GIUSEPPE PEANO (1858-1932) was probably one of the first to carefully study DEDEKIND's booklet (of 56 pages) *'Was sind und was soll die Zahlen?'* immediately after its publication in January 1888. It inspired him to derive some propositions of elementary arithmetic on the basis of DEDEKIND's fundamental conditions (α), (β), (γ), (δ) in the purely formal language he himself developed, using his logic calculus in *a purely syntactic way.* He published the results in his essay '*Arithmetices Principia nova methodo exposita',* also as a small booklet (of 36 pages), one year later in Turin, (op. cit., 1889). In the preface, PEANO expressly pointed out that

> *"Utilius quoque mihi fuit recens scriptum: R. Dedekind, Was sind und was soll die Zahlen, Braunschweig, 1888, in quo quaestiones, quae ad numerorum fundamenta pertinent, acute examinantur".*
>
> [The recently published book by R. DEDEKIND: *»What are the numbers and what are they meant for?«*, in which questions pertaining to the foundations of the concept of number are examined in detail, was very useful.]

It is not entirely incidental to point out that DEDEKIND did not refer to the fundamental conditions (α), (β), (γ), (δ) as *'axioms'* anywhere in his writing. He wrote (in #71) only of *'conditions'* that belong to the definition of the concept of a 'simply infinite system'. We have named them here, in reference to DEDEKIND*'s 'Algebra-Vorlesungen'* of WS 1856/1857 & WS 1857/1858 (op. cit., article 6, p. 67), '*fundamental conditions'* (*"Fundamentalbedingungen"*), because, in these algebra lectures, DEDEKIND did not speak of 'axioms' of group theory either, but only of 'fundamental conditions' of the group concept.

DEDEKIND wanted to show, in his booklet *'Was sind und was sollen die Zahlen?'*, that arithmetic of natural numbers is a *'part of logic'* (as he wrote in the preface, p. VII), and, hence, can be constructed *without any preconditions* (cf. our Chapter 7) and is only *'a direct result of the pure laws of thought'* (*"ein unmittelbarer Ausfluß der reinen Denkgesetze"*). In this respect, the conditions (α), (β), (γ), (δ) have never had the status of axioms or

postulates (cf. Chapter 4) in DEDEKIND's work: they are *conditions* with which certain structured sets are defined.

PEANO, on the other hand, understood them as *axioms*. He copied the four *'fundamental conditions'* of DEDEKIND and wrote them down in the symbolic language that he had invented.

Condition (α) requires that the class of successors n' of elements n of N be a subclass of N. In DEDEKIND's notation, this is expressed in the form $N' \subseteq N$. In PEANO's notation, this is axiom 6, and is written in the form $n \in \mathrm{N} \Longrightarrow n+1 \in \mathrm{N}$. (The notation is slightly different, since DEDEKIND uses the 'less-than' symbol '$<$' fancifully written down with curly ends, instead of the modern symbol $\subseteq$ for set-inclusion, and PEANO uses an inverted 'C' (symbolising 'continetur', or 'deducitur') instead of an implication-arrow '$\Longrightarrow$'.)

Condition (β) is the principle of mathematical induction, and is rewritten by PEANO as axiom 9.

Condition (γ) is rewritten by PEANO as axiom 8 and has, in his notation, the form $n \in N \Longrightarrow n+1 \neq 1$ (again, with an inverted 'C' instead of an arrow).

Condition (δ) expresses the fact that the successor function is one-to-one, which is axiom 7 in PEANO's axiom system: *$a+1=b+1$ is valid if and only if $a=b$ is valid*.

The other axioms in PEANO's axiom system are tautologies and deduction rules from pure logic. They are necessary in a formal calculus in order to fix the rules for proofs.

It is obvious that the axiom system presented by PEANO is nothing more than a mere translation of DEDEKIND's fundamental conditions (α), (β), (γ), (δ) into PEANO's notation. It is also obvious that the fundamental ideas, which governed DEDEKIND's conception of founding the number system, namely, to avoid +, – and ·, and to choose only the successor-operation so that the validity of the principle of mathematical induction becomes logically equivalent to the algebraic property of *'simplicity'*, have been copied by PEANO almost word for word.

BERTRAND RUSSELL, in his work on *'The Principles of Mathematics'* (Cambridge, 1903, Part II, Chapter XIV, §120), discussed at length the axioms of arithmetic that he attributed to PEANO without mentioning DEDEKIND at all. Since RUSSELL's book had an enormous influence in the English-speaking and the French-speaking worlds at that time, it became usual to ignore DEDEKIND's brilliant contribution to the foundation of arithmetic and attribute DEDEKIND's contribution to PEANO.[8] Fortunately, in recent decades, it has become usual also to give credit to DEDEKIND's contribution and to call the axiomatic foundation of arithmetic 'DEDEKIND-PEANO-Arithmetic', DPA.

[8] LOUIS COUTURAT, in his *'Les Principes des Mathématiques'* (Paris, 1905, pp. 54-61), followed RUSSELL in his presentation of the foundations of arithmetic and attributed DEDEKIND's fundamental conditions to PEANO, without mentioning DEDEKIND. Moreover, COUTURAT also presented RUSSELL's definitions of cardinal numbers as collections of all equipotent sets. Neither RUSSELL nor COUTURAT realized that such an introduction of numbers leads to inconsistencies - cf. Chapter 14, section 14.3 & Chapter 15, footnote 8.

DEDEKIND was concerned with the foundation of the *'science of numbers'* (*"Zahlentheorie"*) and, hence, also with the question of what the *subject* of this science is, what numbers really *'are'*, and from which sources one can draw if one wants to prove number-theoretical theorems. In his opinion, these sources lie in the field of logic, in which, according to the understanding of the time, the principles of set formation were also counted.

PEANO refrained from giving reasons for the choice of these axioms. As a starting point, he chose, without any comments, the fundamental conditions established by DEDEKIND. For PEANO, the source from which one can draw, when one wants to do arithmetic, is the axiomatic system, which, at present time, is called DPA.

19.7 The emergence of Structuralism

In the Introduction (pp. 1-6), we posed the fundamental question, in which sense do the objects of the various mathematical objects exist. In the previous chapters, we have seen quite a number of different approaches to the answer to this seemingly difficult question.

A novel answer - at least to the question of what the (natural) numbers *are* - was given by DEDEKIND, as we have seen above.

DEDEKIND's answer was that numbers may not exist as objects that are numbers by themselves, but that they at least exist as objects that belong to a structured system (or set) of objects that are 'numbers' *in the sense of the structured system*. Thus, if it is permitted to use Aristotelian terminology here, we may say that numbers may not exist *kath' hauto* (καθ' αὑτό, *per se*), but that they exist at least *kata symbebêkos* (κατὰ συμβεβηκός, *per accidens*).

DEDEKIND's answer impressed DAVID HILBERT deeply. He recognized that the old question of what the objects of geometry really *are* can be answered in a similar way**:** *no thing is per se a point (without extension) and no thing is per se a straight line (without breadth and of unlimited length); but a thing can be taken for an 'extensionless point' or a 'breadthless line of unlimited length' if it belongs to a system (more precisely, a structured set) of things that, all together, satisfy EUCLID's axioms of geometry*.[9] These things may not be 'extensionless' in the absolute sense, or 'breadthless' and 'straight', or of 'unlimited length' in the absolute sense, but they may have these properties in the sense of the structured set, considered as an artificial universe of objects. These insights, together with a number of others, led him to write his famous essay *'Grundlagen der Geometrie'* in 1899. We will report on this in more detail in Chapter 20.

An adequate terminology, which allows us to formulate the existence of objects that have certain properties only relative to a structured set (or: only in the sense of a structured set), had at that time not yet been found. In Germany, the terminology varied among *'System', 'Inbegriff'* and *'Gefüge'*. In other European languages based in Latin, the terminology of *'structure',* or *'structured set'*, became common from 1935 onward (ÖYSTEIN ORE 1935, MARC KRASNER 1938, NICOLAS BOURBAKI 1939 and others). Only

[9] The axioms, however, must be considered in an *abstract* version, with empty signs for the basic concepts, so that they can be interpreted by things that are not necessarily extensionless in the absolute sense, etc.

after 1950 did the word *'structure'* become generally accepted and, from then on, it even became one of the central concepts of *'modern mathematics'*.[10]

The word *'structure'* is borrowed from Latin. *Struere* means 'to layer', 'to fit together', 'to erect', and *structura* is *'the structure', 'the masonry', 'the orderly joining together', 'the con-struction'*. The word *'structure'* therefore has an architectural meaning in classical Latin.[11] In the Middle Ages, it was occasionally used as a metaphor (cf. G. LIEBERG, op. cit.). But it was only at the end of the 19th century that it acquired the extended meaning (in philosophy, psychology, sociology, linguistics, and also in mathematics) *that the elements and the parts of a system of objects (as in a masonry) relate to each other, and that these relationships are essential for the individual elements and their properties*.

In mathematics, the concept of a *'structured set'* (or, simply, a *'structure'*) can be defined as follows:

Definition. A *structure* is a quadruple consisting of the following four things:

(i) firstly, a non-empty set M (the so-called *universe* of the structure),
 - and for pairwise disjoint index sets I, J and K:
(ii) secondly, a family of unary or n_i-ary functions f_i (for $i \in I$), which are defined for all elements of M and whose values are elements of M,
(iii) thirdly, a family of relations R_j (for $j \in J$) on M, and
(iv) fourthly, a family of elements c_k (for $k \in K$) belonging to M.

Such a structure $\mathfrak{M}$ is (usually) denoted as follows:

$$\mathfrak{M} = \langle M, \{f_i; i \in I\}, \{R_j; j \in J\}, \{c_k; k \in K\} \rangle.$$

The natural numbers are - as we have seen - the elements of the universe M of a simply infinite DEDEKIND-triple $\langle M, \varphi, 1 \rangle$, which can be understood as a structure in which the DEDEKIND-PEANO-axioms are valid (whereby, for the sake of simplicity, we have omitted the curly brackets around the one-element sets $\{\varphi\}$ and $\{1\}$).

The real numbers are the elements of the universe of a structure of the form

$$\mathfrak{R} = \langle \mathbb{R}, +, -, \cdot, \leq, 0, 1 \rangle,$$

in which the axioms of continuously-ordered, real-closed fields are true (RICHARD

[10] At the end of the 19th century, people spoke occasionally of the *structure* of a group (for example, ELI CARTAN in 1894, LEONARD EUGENE DICKSON in 1900) in order to indicate the way in which the group is composed of subgroups and homomorphic images. Compare H. BURKHARDT - L. MAURER: *'Groupes des Transformations continus'*, Encyclopédie des Sciences Math. II,2 (Paris-Leipzig 1916), pp. 161-240, there p. 181. But groups themselves were not yet called 'structures'.

[11] Note that the term *"Fachwerk"* (framework), which KANT and HILBERT liked to use, also means something that comes close to the meaning of the word *'structure'* and also originates from architecture (cf. Chapter 20). They spoke of a 'framework of concepts' (*"Fachwerk der Begriffe"*) - cf. Chapter 20.

DEDEKIND 1872, DAVID HILBERT 1900, op. cit., and EMIL ARTIN & OTTO SCHREIER, 1925). Concrete examples (or 'prototypes') of such structures are

(1.) the continuous hull of the ordered field of rational numbers (constructed using DEDEKIND cuts, DEDEKIND 1872) and

(2.) the set of all equivalence classes of CAUCHY *sequences of* rational numbers (CANTOR, 1872, cf. *'Gesammelte Abhandlungen',* pp. 93-95).

All continuously-ordered, real-closed fields are isomorphic structures, and thus have the same *'abstract type',* and this type *is* the continuum of real numbers.

The imaginary and complex numbers are the elements of the universe of a structure of the form $\mathfrak{C} = \langle \mathbb{C}, +, -, \cdot, 0, 1, i \rangle$, in which the axioms for algebraically closed fields of characteristic 0 are valid and which have the cardinality $2^{\aleph_0}$ (ERNST STEINITZ, 1910). There are also many possibilities to specify such fields concretely, for example, according to CARL FRIEDRICH GAUSS (1811, 1828) and WILLIAM ROWAN HAMILTON (1835):

$$\mathbb{C} = \{(x, y);\ x, y \in \mathbb{R}\}.$$

If we multiply ordered pairs (a,b) according to the rule

$$(a, b) \cdot (c, d) = (ac - bd, ad + bc),$$

then the ordered pair $(0,1)$ is an element whose 'square' plays the role of -1 (where $(1, 0)$ is identified with 1 and $(-1, 0)$ with -1). However, other constructions are also possible, such as the one that goes back to AUGUSTIN-LOUIS CAUCHY (1789-1857). In 1845, he constructed the field of complex numbers as follows[12]:

$$\mathbb{C} = \mathbb{R}[x]/(x^2+1),$$

where $(x^2 + 1) = \{x^2 \cdot g(x) + g(x);\ g(x) \in \mathbb{R}[x]\}$ is the *ideal* [13] in the polynomial ring $\mathbb{R}[x]$ generated by the polynomial $x^2 + 1$. In this domain, the residue-class $x+(x^2 + 1)$ is an element whose square plays the role of –1. Both constructions provide isomorphic fields, because the residue class of the polynomial $a + bx$ can be assigned to the element (a,b).

– * –

In all of these examples of rings and fields of numbers, as well as in HILBERT's model for Euclidean geometry (which will be discussed in Chapter 20 in some detail), it became clear that, for example, the properties

- to be the *'number 3'*, or
- to be a *'negative number'*, or
- to be a *'square root of -1',* or
- to be a *'breadthless line'*, etc.,

[12] A historical note: CAUCHY did not form residual classes, but was content to calculate in congruences, following the example of CARL FRIEDRICH GAUSS.

[13] Notice that the brackets around the polynomial x^2+1 do not have the usual meaning of putting something in parentheses, but have the special meaning of forming the 'ideal' (in the sense of ring theory) generated by the polynomial x^2+1.

are properties that objects will not possess *per se* (*sui generis*), but are properties that objects may possess *per accidens*, i.e., relative to a universe of discourse to which the object belongs. Since around 1950, the view that an object possesses a non-trivial mathematical property if it belongs to a suitable structure and has the property in question relative to that structure is called *structuralism.* The main thesis of structuralism is that one can only refer to mathematical objects, i.e., to objects that have certain mathematical properties, if one also specifies the structures to which these objects belong.

CHARLES PARSONS (op. cit., p. 303) described *structuralism*, in 1990, in a similar way as follows:

> *„By the »structuralist view« of mathematical objects, I mean the view that reference to mathematical objects is always in the context of some background structure, and that the objects involved have no more to them than can be expressed in terms of the basic relations of the structure.“*

19.8 A new approach to Abstraction

The structuralist point of view offered a convincing explication of the *mode of being* of mathematical objects. But the introduction of structures also offers the possibility to deal with abstractions in a new and convincing way.

In the classical theories of abstraction (ARISTOTLE, JOHN LOCKE et al.), many properties of an object are *'taken away'*, in which it is usually unclear as to what, in the end, has actually been taken away and disappeared and what has supposedly been left behind. The quite slow vanishing of the cat, *"beginning with the end of the tail and ending with the grin, which remained some time after the rest of it had gone"* (in LEWIS CARROLL's *'Alice's Adventures in Wonderland'*), is a beautiful, although ironic, example for such an unclear act of abstraction (cf. Chapter 13, section 13.5).

However, by adoption of the structuralist point of view, *the process of abstraction* is more concrete. This is so, since it is not so easy to make explicit all that what should be abstracted, than to list all that what should not be abstracted. That what should not be abstracted is explicitely listed in the structure, namely certain interdependencies (expressed with the help of functions and relations) and names of certain objects.

This approach to abstraction does not define abstraction 'from above' but 'from below'. This change of view makes 'abstraction' much better controllable.

In the examples that we discussed in section 19.7, the structures were always models of certain axiom systems written down in certain *formal languages*. These languages contain names for the functions, relations and special objects displayed in the structure. However, all these names are *'empty signs'* (cf. section 18.2 in Chapter 18). In *colloquial languages,* the names and predicates all have (rather) fixed meanings. In formal languages, however, all of those signs that do not belong to logic are 'empty signs', since their meanings are not fixed in advance. This is the reason why they are called *'formal languages'.* The 'empty signs' get their meanings by stipulating that they denote the corresponding functions, relations and objects displayed in the structure.

All of this which we just elaborated is nothing more than an explicit description of that which DEDEKIND probably had in mind when he described his proof of theorem #134 (cf.

section 19.4), but could not write down because the relevant terminology was not yet available.

Notice that the introduction of *'empty signs'* expresses a kind of *abstraction* in the Aristotelian sense, since all (!) possible meanings are 'taken away' (or, more precisely: are explicitly neglected, ignored). A system of axioms is used to supply those meanings that the named functions, relations and objects should possess *accidentally.*

In this way, the mentally performed acts of abstraction, which had been used in former times, are avoided and replaced with a kind of abstraction that is completely controllable, as is always required in mathematics.

The idea of using empty signs and axioms in order to declare precisely *to what extent abstraction is carried out* is credited to a number of persons, including JOHANN HEINRICH LAMBERT, RICHARD DEDEKIND, HERMANN WIENER, GINO FANO and, most prominently, DAVID HILBERT. It found its clearest exposition in the formulation of HILBERT, in his essay on the foundations of geometry (1899):

> *"Wir denken drei verschiedene Systeme von Dingen: die Dinge des ersten Systems nennen wir Punkte und bezeichnen sie mit A,B,C,...; die Dinge des zweiten Systems nennen wir Gerade und bezeichnen sie mit a,b,c,... ;* etc."
>
> ['We think three different systems of things: the things of the first system we call points and denote them *A,B,C*,...; the things of the second system we call straight lines and denote them a,b,c,...;.etc.']

What 'points' and 'straight lines' *are*, according to our knowledge from daily life, is used in the process of setting up the axioms (cf. Chapter 18), but, as soon as the axioms are formulated, only those contents that are contained in the axioms are allowed to be used in proofs. Therefore, the predicates *'points'* and *'straight lines'*, etc., are introduced as *'empty signs'* and all that is relevant in establishing geometric facts is incorporated into the axioms of geometry.

We will come back to this emergence of the idea *of 'empty signs'*, of an *'implicit definition'* and the new understanding of the concept of an *'axiom system'* in Chapter 20, in the discussion of HILBERT's work on the *'Grundlagen der Geometrie'* (1899).

19.9 The 'abstract' direction in Algebra

The taking of the *'abstract direction'* has led to a paradigm shift that affected the 'nature' of mathematical objects. The old questions about what the different numbers and the different geometric objects *'are'*, *where they exist* and *in what sense they exist* have been answered in a new way in structuralism: all of these objects exist only within structures (in the technical sense described above) and they always have the respective properties only *as* elements of these structures.

HENRI POINCARÉ (1854-1912) described the new 'abstract direction' as follows:

> *"Les mathématiques n'étudient pas des objets, mais des relations entre les objets; il leur est donc indifférent de remplacer ces objets par d'autres, pourvu que les relations ne changent pas. La matière ne leur importe pas, la forme seule les intéresse."*
>
> [Mathematicians do not study objects, but relationships between objects; they do not care

whether these objects are replaced by others or not, if only the relationships remain unchanged. The object is indifferent to them, the form alone is of interest to them."] [14]

HENRI POINCARÉ: *'La Science et l'hypothèse',* I,2; Paris 1902.

In his *Obituary for Emmy Noether* (*'Nachruf auf Emmy Noether',* op. cit., 1935), B. L. VAN DER WAERDEN (1903-1996) wrote, in the same sense:

"Alle Beziehungen zwischen Zahlen, Funktionen und Operationen werden erst dann durchsichtig, verallgemeinerungsfähig und wirklich fruchtbar, wenn sie von ihren besonderen Objekten losgelöst und auf allgemeine begriffliche Zusammenhängw zurückgeführt sind."

[All relations between numbers, functions and operations only become transparent, generalizable and really fruitful when they are detached from their particular objects and traced back to general conceptual connections.]

These quotations prove that the 'abstract direction' had already prevailed in the first decades of the 20th century. At the same time, this gave rise to the great new task of examining the *structures* that were now used everywhere more closely and of studying their structure in detail. Structuralism, which actually had a philosophical objective, now led to a *'mathematics of structures'.* Under the deep impression that VAN DER WAERDEN*'s* two-volume work on *'Moderne Algebra'* (1930/1931) had brought about, NICOLAS BOURBAKI began to present the new structuralist thinking in encyclopaedic form from about 1935 on (cf. BOURBAKI, op. cit.).

19.10 Final considerations

Since antiquity, it was common practice to refer to numbers and geometric straight lines, surfaces and solids as *quantities* (see the quotation from ARISTOTLE in Chapter 15, and also compare Chapter 18). Since numbers and geometric objects were the most important objects of mathematics for a long time, mathematics was called the *'science of quantity'.*

This term is no longer appropriate. Since it became apparent, during the 19th and early 20th centuries, *that the fundamental objects of mathematics always exist only as elements of certain structures,* mathematics could rather be called a *'science of structures'.* But even such a designation would not do justice to mathematics as a whole. It is indisputable, however, that the concept of structure has become extremely important not only in the discussion of the foundations of mathematics, but also in almost all areas of mathematics.

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[14] With similar words, DESCARTES had already expressed this thought in the 2nd chapter of his *'Discours de la Méthode'* (1637) - compare R. DESCARTES *Œuvres,* vol. VI, p. 20.

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Chapter 20
HILBERT's critical Philosophy

DAVID HILBERT (1862-1943) was an excellent mathematician who did outstanding work in many areas of mathematics and mathematical physics. He was also one of the few who thought intensively about the foundations of mathematics. Thereby, he became the pioneer of many new developments.

It is striking that the guidelines that HILBERT followed in his research on the foundations of mathematics are perceived very differently. He himself said very little about his philosophical standpoint. But, in a lecture on '*Die Grundlagen der Mathematik*' (in Hamburg, 1927), he spoke of the urgent need for an *'extension of the formal standpoint ... to all mathematics'* (*"eine Ausdehnung des formalen Standpunktes ... auf die gesamte Mathematik"*). These and similar statements have probably led to the fact that, for many authors, even today, HILBERT is still the *arch-father of formalism*. But many have also vehemently protested against this assessment and tried to make it clear that HILBERT, outside of his metamathematics, can by no means be classified as one of the pure *'formalists'*. JAAKO HINTIKKA, for example, wrote (op. cit. p. 15):

> *„Hilbert's philosophy of mathematics is almost universally labeled »formalism«, ... [but] such a classification is highly misleading."*

Indeed, HILBERT himself often enough pointed out that, when building a mathematical theory, the contentual point of view should be linked to the formal point of view, as we have already discussed in Chapter 18.

Some have preferred to call HILBERT a *'logicist'*, or even one who, at least at times, professed *'constructivism'*, and even *'finitism'* (JOSÉ FERREIRÓS, op. cit., WILFRIED SIEG, op. cit.). Still others preferred to call him a *'structuralist'* (CHARLES PARSONS, op. cit., STEWART SHAPIRO, op. cit., p. 165, and others). It seems to be difficult to give a correct statement of the position taken by HILBERT.

But, for HILBERT, none of these positions played a major role. He approached them occasionally, but he was not committed to any of them. He did not join any of them.

We do not believe that any of the above positions apply to HILBERT without reservation. The only characterisation that is really true is probably the description of the general purpose of his research, which could be described as *'critical philosophy'* (*'Kritizismus'* in German). In reference to IMMANUEL KANT (1724-1804), this refers to the doctrine that aims

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to explore the *'possibilities and limits'* of a theory or a construct of thoughts - here, a mathematical theory. Although HILBERT did not subscribe to the philosophical views advocated by KANT, he did subscribe to the general objective of KANT's *'critical philosophy'*.[1] In carrying out his *'critical'* concerns, HILBERT came close to some of the above-mentioned positions. But these positions are subordinate to his *critical philosophy*. This will be explained in the following.

20.1 HILBERT's Philosophy of Mathematics

We learn a lot about HILBERT*s* philosophy of mathematics if we start from his motto:

> *"Da ist ein Problem, suche die Lösung. Du kannst sie durch reines Denken finden; denn in der Mathematik gibr es kein Ignorabimus!"*
>
> [There's a problem, look for the solution. You can find it by pure thinking; for in mathematics there is no Ignorabimus!"]
>
> DAVID HILBERT in his lecture on August 8, 1900 in Paris during the Second International Congress of Mathematicians, cf. Hilbert's *Gesammelte Abhandlungen*, Volume III, Berlin 1935, pp. 290-329.

The postscript that *'there is no ignorabimus in mathematics'* is a replica of a claim by the physiologist EMIL DU BOIS-REYMOND (1818-1896) that was well known and often quoted at that time.[2] HILBERT concluded his lecture on '*Naturerkennen und Logik*' , which he gave in Königsberg/Prussia in 1930, with the words

> *"Statt des törichten Ignorabimus heiße im Gegenteil unsere Losung: Wir müssen wissen, Wir werden wissen.*
>
> [Instead of the foolish »we will never know«, let our motto be: We must know, we will know"]

This inspiring exclamation should perhaps remind us of the optimistic *"Sapere aude"* of HORACE (Epistle I,2, verse 40), which KANT translated, in his famous essay *'Beantwortung der Frage: Was ist Aufklärung',* 1784), with: *"Habe Mut, dich deines eigenen Verstandes zu bedienen"* (*'Have courage to use your own understanding'*).

In his Göttingen lecture '*Wissen und Mathematisches Denken'* (1922/1923, pp. 23-24), HILBERT expressed his opinion a little more clearly as follows:

> *"... daß ein jedes klar formulierte mathematische Problem einer vollen Erledigung fähig*

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[1] See, for example, his lecture '*Wissen und Mathematisches Denken*', op. cit., pp. 87-90, or the lecture '*Naturerkennen und Logik*' (1930), HILBERT*'s* '*Gesammelte Abhandlungen*', vol. III, pp. 378-387, especially p. 383. See also HILBERT*'s* lecture '*Über das Unendliche*' (1925), where he joins KANT. The meaning of the term *'critical philosophy'* is explained in Chapter 12, section 12.2.

[2] EMIL DU BOIS-REYMOND (the brother of the mathematician PAUL) claimed, in 1872 (published in 1881), that there are unsolvable problems in the natural sciences. He said that, in relation to the riddles of what is matter and what is force, natural scientists must admit *'with male renunciation'*: *"Ignoramus et Ignorabimus"* (*we do not know and will never know*). The saying by DU BOIS-REYMOND became so popular that even the poet ARNO HOLZ (1863-1929) wrote a play, in 1912, entitled '*Ignorabimus*', which deals with the unrecognizability of occult powers. Among the characters in the drama is a certain Prof. Dr. Dufroy-Regnier; the similarity with the name "Du Bois-Reymond" is, of course, not accidental.

sein müsse, sei es, daß es gelingt, die gestellte Frage im positiven Sinne zu beantworten, sei es, daß die Unmöglichkeit der gestellten Forderung und damit die Notwendigkeit des Mißlingens aller Lösungsversuche dargetan wird."

[... that every clearly formulated mathematical problem must be capable of a full solution, whether it succeeds in answering the question posed in a positive sense, or whether the impossibility of the requirement posed and thus the necessity of the failure of all attempts at solution is demonstrated.]

In order to be able to *'completely solve'* a mathematical problem, one must have a good knowledge of *the possibilities and limitations* of the theory from which the problem originates. To do this, the theory must be structured in such a way that it can itself become the object of a mathematical investigation (i.e., a *'metamathematical'* investigation, as HILBERT described it from about 1922 onwards). HILBERT thought that the most important mathematical disciplines were now mature enough to be put into a form in which they could be investigated by *metamathematical* methods.

In Chapter 17, we already referred to a statement by HILBERT, intended for the Mathematicians Congress in Chicago in 1893, that states:

"In der Geschichte einer mathematischen Theorie lassen sich meist drei Entwicklungsperioden leicht und deutlich unterscheiden. Die naive, die formale und die kritische."

[In the history of a mathematical theory, three periods of development can usually be easily and clearly distinguished. The naive, the formal and the critical.]

In the case of the theory of algebraic invariants, HILBERT had clearly distinguished the three periods of development in his above-mentioned essay.

In the case of algebra, these three periods can be described roughly as follows: The period from DIOPHANT to LEONARDO OF PISA may be seen as the *'naive period'*, in which the joy of a mathematical treatment of simple, concrete problems predominated. In the period that followed, in which it was a matter of establishing general, formal laws (solution formulae for algebraic equations of the n^{th} degree [ALKHWOÁRIZMI, SCIPIONE DAL FERRO, GERONIMO CARDANO, FRANÇOIS VIÈTE and others], solution formulae for linear systems of equations [GOTTFRIED WILHELM LEIBNIZ, GABRIEL CRAMER], etc.), one can recognize the second 'formal period'. The third 'critical period' began at the beginning of the 19th century, with the contributions of PAOLO RUFFINI (1765-1822), NIELS HENRIK ABEL (1802-1829) and ÉVARISTE GALOIS (1811-1832), when the *possibilities and limits* of algebraic methods were discussed.

HILBERT spoke of the *'critical period of development'* and referred (without mentioning the name) to IMMANUEL KANT.[3] He was probably inspired, in particular, by the following statement by KANT (KrV A298, B355), which also speaks of a similar three-stage development:

"Alle unsere Erkenntnis hebt von den Sinnen an, geht von da zum Verstande und endigt bei der Vernunft."

[3] HILBERT was well acquainted with KANT*'s* works. For example, in a disputation (on October 11, 1884), which, at that time, was part of the doctoral procedure in Königsberg, he chose a topic from KANT*'s* philosophy.

> [All our knowledge is raised from the senses, goes from there to understanding, and ends at reason.]

An almost identical statement by KANT (KrV: A702, B733) is the motto of HILBERT's book on the *'Foundations of Geometry'*:

> *"So fängt denn alle menschliche Erkenntnis mit Anschauungen an, geht von da zu Begriffen und endigt mit Ideen."*
>
> [Thus all human knowledge begins with views, goes from there to concepts and ends with ideas.]

The two quotations are taken from KANT's *'Kritik der reinen Vernunft'*. In this work, KANT wanted to explore the *possibilities and the limits* of pure reason (KANT, KrV: A XII). Since then, the doctrine that aims at such an exploration has been called *'Kritizismus'* in German; rendered into English, this means *'critical Philosophy'* (cf. Chapter 12). Since HILBERT was concerned with the exploration of the *possibilities and the limits of* mathematical theories, one may adopt the term that has become common and speak of *'critical Philosophy'* in HILBERT*'s* case as well.

HILBERT's critical Philosophy is generally concerned with the solution of logical, epistemological and ontological problems posed by a mathematical theory. All three kinds of problems are to be solved in such a way that, at the same time, *the possibilities and limits* of the theory become manageable and can be investigated by mathematical means. For this to succeed, the theory must be constructed, in a very strict sense, from a formal point of view (cf. Chapter 18). HILBERT first demonstrated that this can indeed be achieved using geometry as an example.

20.2 HILBERT's axiomatization of Geometry

HILBERT published his essay on *'Grundlagen der Geometrie'* (op. cit.) in 1899. As he pointed out in the epilogue, he wanted to give *'a critical study of the principles of geometry'* (*"eine kritische Untersuchung der Prinzipien der Geometrie"*). - Again, we notice the use of the attribute *'critical'*, which, here too, was meant as an allusion to KANT*'s 'critical philosophy'*. - The book is not, therefore, about developing geometry from scratch, for example, in the form of a textbook of the type of EUCLID's *'Elements',* ALPHONSO BORELLI's *'Euclides restitutus'* (1679), or LEGENDRES' *'Éléments de Géométrie'* (1794), but about establishing classical geometry in such a way that its *possibilities and limits* can be explored. But HILBERT also wants, as he wrote in the introduction, to establish geometry in such a way that, with its help, a *'logical analysis of our spatial perception'* will be possible.

The anchoring of geometry in reality is thus acknowledged. But a *'logical analysis of our spatial perception'* can only succeed if geometry is constructed as a *'framework of concepts'* (*"Fachwerk von Begriffen"*) [4] - cf. HILBERT*'s* letter of December 29, 1899, to FREGE, op. cit.

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[4] HILBERT adopted KANT's term *"Fachwerk von Begriffen"* and used it frequently from 1894 onwards (see I. KANT: *'Anthropologie in pragmatischer Hinsicht',* Akademie-Ausgabe der Werke KANT*'s*, Volume 7 (Berlin 1907), p. 184).

For HILBERT, geometry as a *'general field of knowledge'* (*"als allgemeines Wissensgebiet"*) is a *'perfect natural science'* (as stated in the manuscript of HILBERT 's lecture '*Elemente der Geometrie*', March 1899, p. 1, prepared by H. v. SCHAPER), but the *theory* of the field of knowledge is the *'framework of its concepts'*, which is a field of pure mathematics (cf. HILBERT 'Ges. Abhandlungen', op.cit., Vol. III, p. 148). Thus, if geometry is considered merely as a *'general field of knowledge'*, it can only be based on individual empirically gained insights and is hardly able to provide universally valid statements. If, however, it is treated as a field of pure mathematics, then it is based on idealizations that make it possible to prove universally valid propositions on the basis of axioms. Geometry is then treated purely conceptually, and it is possible to make statements about *all* objects that fall under the respective concepts.

But, on the other hand, it is often the case that, in a geometry that is considered as a *'general field of knowledge'*, i.e., as a natural science, much can be grasped at a glance through sensory perception. But not everything that can be determined by visual inspection can be understood in the theory of the field of knowledge by logical progression from concept to concept. The *straightness* of a line, for example, can be easily perceived with the senses, but cannot be described with elementary geometric concepts alone. PROKLUS (ca. 411-485) remarked, in his '*Commentary on Euclid's Elements*,' that a lot of facts are *'clear to the eye',* but quite often are very difficult to analyze and to explain *'for scientific thinking'*. So, HILBERT'*s* aim is to give a foundation for geometry as a scientific theory in such a way that it becomes a *framework of concepts*, so that, with its help, a *'logical analysis of our spatial view'* becomes possible.

The problem arises of how to introduce the *basic concepts* of geometry so that all these wishes can be fulfilled. Explicit definitions of the basic concepts do not seem to exist, as has been known since DESCARTES, PASCAL, HOBBES, TSCHIRNHAUS and others (cf. Chapter 10). So, HILBERT must get along without definitions of the basic concepts. He therefore uses (similar to JOHANN HEINRICH LAMBERT and MORITZ PASCH, see Chapter 18) *'empty signs'* to signify the basic concepts. But these empty signs are indirectly filled with meanings, because they are woven into the axioms. HILBERT thus chooses an axiomatic treatment of geometry. He begins with the winged words:

> *"**Erklärung**. Wir denken drei verschiedene Systeme von Dingen: die Dinge des ersten Systems nennen wir Punkte und bezeichnen sie mit A, B, C, ...; die Dinge des zweiten Systems nennen wir Gerade und bezeichnen sie mit a, b, c, ...; die Dinge des dritten Systems nennen wir Ebenen und bezeichnen sie mit α, β, γ, ...; (...) Wir denken die Punkte, Geraden, Ebenen in gewissen gegenseitigen Beziehungen und bezeichnen diese Beziehungen durch die Worte wie »liegen«, »zwischen«, »parallel«, »kongruent«, »stetig«; die genaue und für mathematische Zwecke vollständige Beschreibung dieser Beziehungen erfolgt durch die Axiome der Geometrie."*
>
> [**Declaration**. We are thinking of three different systems of things: the things of the first system we call *points* and call them *A,B,C,...;* the things of the second system we call *straight lines* and call them *a,b,c,...;* the things of the third system we call *planes* and call them *α,β,γ,....* [...] We think of points, lines, planes in certain mutual relations and denote the relations by words like *"to be placed on", "between", "congruent"*; the exact and, for mathematical purposes, complete description of these relations is given by the axioms of geometry. [etc.]

In speaking of *'systems of things'*, HILBERT joined the terminology of DEDEKIND ('*Was sind und was sollen die Zahlen?*', 1888, #1). It is thus emphasized that the basic terms are called up as empty terms (i.e., merely as unspecified, abstract *'things'*) and only the axioms that follow give details of their intended meaning and significance. JOHANN HEINRICH LAMBERT (1766), HERMANN WIENER (1891, op. cit.) and GINO FANO (1892, op. cit.) had already expressed themselves in a similar way. LAMBERT wrote in §11 of his essay *'Theorie der Parallellinien'* from 1766, which was published 1786 posthumously (op. cit., but notice that the term *'symbolic'* is understood there in the sense of *'formal with empty signs'*):

> *"Und da Euclid's Postulata und* (die) *übrigen Grundsätze* (...) *mit Worten ausgedrückt sind: so kann und soll gefordert werden, dass man sich in dem Beweise nirgends auf die Sache selbst berufe, sondern den Beweis durchaus symbolisch vortrage - wenn er möglich ist. In dieser Absicht sind Euclid's Postulate gleichsam wie eben so viele algebraische Gleichungen, die man bereits vor sich hat, und aus welchen x, y, z, etc. herausgebracht werden soll, ohne dass man auf die Sache selbst zurück sehe."*
>
> [And since Euclid's postulates and the other principles are expressed in words, it can and should be demanded that nowhere in the proof one refers to the matter itself, but presents the proof quite symbolically - if possible. In this intention, Euclid's postulates are like so many algebraic equations that one already has in front of oneself, and from which *x, y, z,* etc. should be brought out, without looking back to the thing itself.]

WIENER wrote (op. cit., p. 46) in the same spirit:

> *"Die Objekte* [der projektiven Geometrie der Ebene] *seien Punkte und Geraden, (...) oder, vom geometrischen Gewande losgelöst: es seien Elemente von zweierlei Art vorausgesetzt, und zweierlei Operationen (...)."*
>
> [The objects [of projective geometry of the plane] are points and straight lines, ... or, detached from the geometrical garment: elements of two kinds are presupposed, and two kinds of operations, ...]

and FANO (op. cit., p. 108) wrote:

> *"A base del nostro studio noi mettiamo una varietà qualisiasi di enti di qualunque natura; enti che chiameremo, per brevità, punti, (...)"*
>
> [On the basis of our study we put a variety of entities of any nature; entities that we will call, for brevity, points, ...]

HILBERT probably knew the essays of LAMBERT, WIENER and FANO and willingly adopted their style of expression. Even sharper than HANKEL and PASCH (cf. Chapter 18), this way that DEDEKIND, LAMBERT, WIENER and FANO choose to convey their ideas is supposed to express that a previous knowledge of what *points, lines* and *surfaces* 'are' is not assumed in the axiomatic treatment of geometry. In this way, the objects of geometry are introduced without any need for abstraction (cf. Chapters 3 and 19) !

So, HILBERT wants to talk about three different systems of things and three different relations between these things. He does not want to define in advance what *'points', 'straight lines', 'planes'* are by their nature and what the relations of *'incidence',* '*betweenness'* and *'congruence'* mean. The words are used as if they were meaningless, *'empty'* words. Everything that one needs to know about their meanings should result from

the axioms that have been established. These axioms may not fully grasp the *intended contents* of the basic concepts, but they can indicate what is meant by them and what is relevant from a mathematical point of view. In this respect, it is said that the axioms *implicitly define* the basic concepts.[5]

In HILBERT*'s* essay, immediately following the above-mentioned *'declaration'*, there is a list of axioms. The axioms of geometry have the following appearance (however, without using the logical symbols ∀, ∃, & for *'for all', 'there exists'* and *'and'),* for example:

Axiom (I.1) $\forall A \forall B \exists a \, (I(A,a) \ \& \ I(B,a))$,

where I is an (empty) sign for a two-place relation whose intended interpretation is the *incidence relation.*[6] So, the axiom has the intended interpretation: *For any two points A and B, there is a straight line a, on which the two points A and B lie.*

A few conclusions are drawn from the axioms; the emphasis is on the *metamathematical* results.

20.3 HILBERT's critical study of Geometry

It is shown that the axioms, except for isomorphism, have only one single model, that the axiom system is therefore *categorical*, and thus, as we can conclude, is also *complete*. Categoricity means that classical geometry is unambiguously determined by the given axiom system (except for isomorphism): it is the usual 3-dimensional geometry over the field of real numbers, ℝ. HILBERT's axiom system thus provides a characterization of the classical geometry of space.

The independence of the postulate of parallels from the other axioms is strictly proven, as is the independence of the axioms of congruence, the independence of the axioms of continuity, the independence of the postulate of ARCHIMEDES, the unprovability of PASCAL's theorem from DESARGUES' theorem, etc. It is also shown that, in the plane, the theorem of DESARGUES is provable with the help of the congruence axioms, but that it is unprovable without the use of these congruence axioms.

All of these results are only possible because, in the formulation of the axioms, nothing was said about the contents of the basic concepts beyond the relations laid down in the axioms. The basic concepts can therefore also be interpreted differently here than usual (as had already been done earlier e.g. by BOOLE in the case of Arithmetic, see Chapter 18). In particular, *'straight'* lines can also be interpreted by such lines that are crooked or bent in the intuitive sense!

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[5] One speaks of *'implicit definitions'* because the (empty) signs are woven or 'folded' (*plicare* (lat.) = to fold) into the axiom system, and the axiom system as a whole *limits* the possibilities of interpreting these signs. JOSEPH-DIEZ GERGONNE (1771-1859) was the first to speak of *'implicit definitions',* cf. his *'Essai sur la Théorie des Définitions'* (Ann. de Math. 9 (1818), p. 1-35). The idea of *'implicit definitions'*, however, goes back to JOHANN HEINRICH LAMBERT, cf. the above citation from his essay *'Theorie der Parallellinien'* from 1766, op. cit.. See also the Epilogue, section E.1, for more details.

[6] In order not to run the risk of using any meanings of words in the proofs that are not part of the preconditions, today, instead of colloquial words, only letters or other empty, meaningless signs are used.

In a letter to FREGE (of December 29, 1899), HILBERT explained the aims of his essay on the '*Foundations of Geometry*' as follows:

> *"Ich bin zu der Aufstellung meines Systems von Axiomen durch die Not gezwungen: ich wollte die Möglichkeit zum Verständnis derjenigen geometrischen Sätze geben, die ich für die wichtigsten Ergebnisse der geometrischen Forschungen halte: dass das Parallelen-axiom keine Folge der übrigen Axiome ist, ebenso das Archimedische etc. Ich wollte die Frage beantworten, ob der Satz, dass in zwei gleichen Rechtecken mit gleicher Grundlinie auch die Seiten gleich sind (dieser Satz ist doch die Grundlage der ganzen Flächenmessung), bewiesen werden kann, oder vielmehr wie bei Euklid ein neues Postulat ist. Ich wollte überhaupt die Möglichkeit schaffen, dass man solche Fragen verstehen und beantworten kann, warum die Winkelsumme im Dreieck 2 Rechte ist und wie diese Thatsache mit dem Parallelenaxiom zusammenhängt. (...) Dies also meine Hauptabsicht."*
>
> [I am forced to set up my system of axioms by necessity: I wanted to give the opportunity to understand those geometrical theorems which I consider to be the most important results of geometrical research: that the postulate of parallels is not a consequence of the other axioms, as is the Archimedean axiom, etc. I wanted to answer the question whether the theorem that in two equal rectangles with equal base lines the sides are also equal (this theorem is - after all - the basis of the whole theory of measurement of surfaces) can be proved, or rather, as in Euclid, is a new postulate. I wanted to create the possibility that one can understand and answer such questions, why the sum of angles in a triangle is 2 rights and how this fact is connected with the postulate of parallels. ... So this is my main intention.]
>
> Extract from a letter from HILBERT to FREGE, reprinted in G. FREGE '*Wissenschaftlicher Briefwechsel',* op. cit., p. 65.

The *'main intention'*, then, according to HILBERT*'s* own confession, was a critical, meta-mathematical study of the principles of geometry.

The critics of HILBERT*'s* writing have regretted that the actually descriptive geometric content of HILBERT's axiomatic theory has been neglected. That points have no extension and lines have no breadth, and that circles are really 'round' and straight lines are really 'straight', are fundamental properties of Euclidean geometry, but seem to play no role in HILBERT's axiomatic theory. It was objected too that geometric objects were deprived of their 'essence' and could no longer *show* themselves in the mind; they were only accessible to calculation and logical derivability.

This is largely correct, for HILBERT's axiom system makes no statement about the nature of geometric objects. Only in a geometry based on diagrams that can be produced with mechanical devices (such as compasses, rulers, angled hooks, etc.) can one rely entirely on the contents given by sense-perception. (In such a pre-Euclidean geometry, the postulate of parallels is valid, apparently). But such a geometry does not deal with points without extension, lines without breadth, etc., nor does it proceed purely conceptually, and thus cannot provide a *'logical analysis of our spatial view'* (see above). HILBERT*'s 'main intention'* described above would not be realizable in such a 'content-based' geometry.

For a detailed discussion of HILBERT's *'Grundlagen der Geometrie*', we refer to our essay in the Jahresbericht of the DMV 115 (2014), op. cit.

20.4 HILBERT's concept of an Axiomatic Theory and his Metamathematics

HILBERT continued his *'critical'* investigations in the following years and decades. In his lecture in Rome in 1908, at the IV International Congress of Mathematicians, he explained his *critical* concern once again and wrote that his aim was to be able to deal with questions of provability or unprovability in mathematical theories. He spoke, in particular, of *'proof-critical investigations'* (*"beweiskritischen Untersuchungen",* HILBERT: '*Gesammelte Abhandlungen*', op. cit., volume III, p. 72). But only from 1917 on did he find enough time to continue his investigations. They led him (in his famous lecture in Zürich on '*Axiomatic Thinking*' (*'Axiomatisches Denken'*) and his 1917/1918 lecture in Göttingen) to a more precise description of *'formal logic'* and the concept of *'axiomatically constructed mathematical theory'*.

For HILBERT*, thinking axiomatically* means, when considering a mathematical theory, concentrating on studying the *framework of concepts* of that theory, with possible interpretations (for example, constructed on a set theoretical basis) being secondary. What is meant by this was explained in some detail above using the example of geometry.

From about 1920 on (cf. HILBERT*'s* '*Gesammelte Abhandlungen*'*,* op. cit., Volume III, p. 202), his investigations led him to a substantial deepening of his previous *critical* investigations, which he [7] called now *'proof-theoretical investigations'* or *'meta-mathematical investigations'* from 1922 on.

PAUL BERNAYS (in his essay '*Thesen und Bemerkungen ...*'*,* op. cit. 1937) pointed out that the

> *"Sonderung von Mathematik und Meta-Mathematik ... der Kantischen Teilung der Philosophie in »System« und »Kritik« nachgebildet (sei)".*
>
> [... separation of mathematics and meta-mathematics ... was modeled after the Kantian division of philosophy into »system« and »critique«.]

This reference to KANT is noteworthy, for it probably originates from HILBERT himself. In the preface to the '*Critik der Urteilskraft'*, Berlin 1790, p. VI, KANT wrote about the *"System der reinen Philosophie"* (*'system of pure philosophy'*) and that

> *"die Kritik den Boden zu diesem* [System oder] *Gebäude vorher so tief (...) erforscht haben* [müsse], *damit es nicht an irgendeinem Teile sinke, welches den Einsturz des Ganzen unvermeidlich nach sich ziehen würde."*
>
> [... the critical philosophy must have explored the ground to this [system or] building so deeply .... beforehand, so that it would not sink to any part, which would inevitably cause the collapse of the whole.]

---

[7] The terms *'metamathematics'* and *'metageometry'* only appeared in literature in the last third of the 19th century. They were used in the sense of *'metaphysics of general spaces'* (meaning *n-dimensional* spaces, non-Euclidean spaces, etc.). HERMANN VON HELMHOLZ (1878), GEORG CANTOR (1884, 1887), EDMUND HUSSERL (1893, see Husserliana volume 40 (2009), p. 25), PAUL NATORP (1910), BERTRAND RUSSELL (1895, 1897), WILHELM WUNDT and others used these terms. With HILBERT*,* however, the word *'metamathematics'* has been given a different meaning, which has already been hinted at above.

HILBERT formulated, as the main problem of his metamathematics, the problem of proving the consistency of the DEDEKIND-PEANO theory (of first order) of natural numbers, DPA. HILBERT, in the years around 1920, had worked out a plan in full generality for arbitrary first-order theories *T* of how to answer the question of consistency.

One must completely formalize the axiomatically constructed theory *T* considered, so that its axioms, formulas and theorems, stripped of all content, are only finite series of signs, and logical reasoning can be represented as a mechanical operation with these series of signs. It is then to be checked whether 'proofs' (i.e., derivations) in *T* can lead to series of signs of the form $0 \neq 0$. Consistency of *T* would then be nothing more than the statement that certain series of signs are not derivable in *T*. This is obviously a statement that also makes sense from a finitist point of view, and should therefore be provable *'on the ground of the finite'* (*"auf dem Boden des Finiten",* cf. HILBERT*'s* '*Gesammelte Abhandlungen*', op. cit., volume III, p. 182), as HILBERT thought.

From 1922 onwards, HILBERT called the completely formalised *'finite approaches'* to mathematical theories (*"die inhaltlichen finiten Betrachtungsweisen"*) *metamathematics* (cf. HILBERT*'s* '*Gesammelte Abhandlungen*', op. cit., Volume III, pp. 184 & pp. 157-177).

The goal HILBERT had in mind was not achievable. But the path to it, which he had cleared, nevertheless led to a series of deep and unexpected results, among which GÖDEL's results are to be counted. It was conjectured by KURT GÖDEL and JOHANN VON NEUMANN in 1930 that the consistency of the arithmetic of natural numbers, with addition and multiplication (!) on the basis of the system DPA of the DEDEKIND-PEANO's axioms (in first-order logic), cannot be proved by finite means. However, GÖDEL never published a proof of this conjecture, and no proof has been found among his posthumous works (cf. JAN VAN PLATO, 2020, op. cit.). The truth of this conjecture was first proved by PAUL BERNAYS and published in volume 2 of the monograph written by DAVID HILBERT and PAUL BERNAYS on the *'Grundlagen der Mathematik'* (Berlin 1939, op. cit.) and, since then, has been called the *second* GÖDEL *incompleteness statement*. It states, more precisely, that the statement $\mathcal{Wid}(\mathsf{DPA})$, which expresses in coded form the consistency ("**Wid**erspruchs-freiheit") of the system DPA, is not provable in the system DPA itself ($\nvdash$ is the crossed-out (i.e., negated) sign for 'derivability'):

$$\mathsf{DPA} \nvdash \mathcal{Wid}(\mathsf{DPA}).$$

This incompleteness theorem shows that *a safeguarding of number theory, and hence also of pure mathematics in general, is impossible with finite methods alone (more precisely: with methods available in the* DPA *system*).

But it has also been shown (by GERHARD GENTZEN, 1936, op. cit.) that, with certain 'insightful' transfinite methods, it is nevertheless possible to carry out consistency-proofs (cf. WOLFRAM POHLERS: *'Proof Theory',* 2009, op. cit., pp. 128-130). GENTZEN's result states (strictly speaking): *if the methods used in meta-theory are free of any contestability, then arithmetic* DPA *is also free of any contestability.*

In conclusion, it should be emphasized that these are results that have arisen on the ground of *critical philosophy*, namely, within a research program that (following KANT) has as its object the exploration of the *possibilities and limits* of a mathematical theory.

This research programme was decisive for HILBERT during his entire time as a university teacher (from 1892/1893 to 1930).

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Epilogue

Ἐν ἀρχῇ ἦν ὁ λογος, ...

In the previous chapters, we have dealt with some fundamental questions in the field of the philosophy of mathematics. In particular, we have dealt with the questions of *how the objects of the various mathematical theories exist, and from what sources we draw when we prove theorems.*

We have presented in detail the most important answers given from antiquity to the early 20th century. Much of what was answered was deeply thought through, but still not always accurate. In particular, it became clear that there does not seem to be a realm of mathematical objects anywhere, neither in the sense-perceivable world nor in a supersensible world.

It is therefore not possible for mathematical theories to be determined by the realms of objects they are supposed to investigate. This also means that the Aristotelian concept of a scientific theory is not adequate for a mathematical theory. The totality of all objects that a mathematical theory should deal with is not a realm of objects that exists somewhere preceding the construction of the theory (cf. Chapter 3, section 3.1). Mathematics is not a (natural) science in the Aristotelian sense (cf. Chapters 3 & 7).

The starting point of a mathematical theory is not a given realm of objects - it is the design of its framework of concepts. Mathematics is not bound to a realm of objects of the real world although most of all mathematical theories have their roots in the investigation of the structure of the real world. In order to be successful in such an investigation, however, it is necessary to subject that part of the world to be examined to sufficient abstraction and idealization beforehand. This also means that a mathematical theory that deals with a certain field of knowledge arises *through a projection* of the individual empirically established facts of the field of knowledge *into a realm of concepts* (cf. DAVID HILBERT: *'Axiomatisches Denken'*, Math. Annalen 78 (1918), pp. 405-415, Reprinted in his *'Gesammelte Abhandlungen',* vol.3 (1935), pp. 146-156).

This insight, however, may be found among mathematicians and philosophers not earlier than at the end of the 18th century. Because of its importance we will have a brief look into the history of the emergence of that what is called a *'concept'*. What are *'concepts'*, how can they be introduced (by which kind of definitions?), and what should they achieve?

E.1 Of Concepts and defining concepts by implicit definitions

The classical understanding of what *concepts* should achieve, may be expressed as follows:

> *With a concept, the mental access to a sensually perceived thing should succeed which would otherwise not be accessible to thought.*

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U. Felgner, *Philosophy of Mathematics in Antiquity and in Modern Times*,
Science Networks. Historical Studies 62, https://doi.org/10.1007/978-3-031-27304-9

This indicates that *concepts* are logical products. Concepts need to have precise definitions. The definition of a concept may not rely upon undefined terms and not on sensual perceptions.

This means in particular, strictly speaking, that the Euclidean ὅροι (definitions) do not define concepts, and that the Cantorian definition of the notion of *'set'* is just as little a concept in the proper sense of the term (cf. Chapters 4 and 15). This was agreed by DESCARTES, PASCAL, ARNAULD, TSCHIRNHAUS, BOREL, HAUSDORFF and many others. They refrained from defining the basic concepts of geometry, arithmetic, set theory etc. This attitude, however, is unsatisfactory as we have argued in Chapter 10, section 10.2. The point is that most of the basic concepts of mathematical theories are not definable on condition that 'definability' refers to defining the *essence* of the basic objects. The concept of *'set'* for example is such a general concept that it can not be defined by even more general concepts. The same applies to the basic concepts of geometry.

If definitions of the essence (*definitiones essentiales* in the sense of ARISTOTLE) are not available then we have to look at other ways of introducing the basic concepts of mathematical theories.

The classical understanding of what *concepts* are was worked out in antiquity. The first step had been undertaken by PLATO by introducing objects which he called *'ideas'.* These *'ideas',* however, do not belong to the sphere of the human intellect and, hence, are not yet that what we want to call *'concepts'* (cf. Chapter 2, section 2.5).

Further steps were taken by the Greek and Roman *Stoic philosophers*. They transformed the Platonic doctrine of *'recollection'* (*anamnesis*) and the doctrine of *'ideas'* into an epistemological theory. The role, which was played by the Platonic *'idea'* was undertaken now by that what was called *'ennoia'* (ἔννοια, notion), *'prolepsis'* (πρόληψις, prenotion, the preliminary grasping), and *'katalepsis'* (κατάληψις, the comprehensive grasping), cf. Chapter 4, section 4.4, and Chapter 8, section 8,6. However, all these terms are not yet carefully enough defined and do not have the same meaning as the word *'concept'* in the classical understanding.

The classical understanding of that what a *'concept'* is, was reached not earlier than about 900 years after PLATO. It was ANICIUS MANLIUS TORQUATUS SEVERINUS BOËTHIUS (he lived ca. 480 - 525) who introduced the term *conceptio animi* and used it as a name for that which allows the mental access mentioned above. That which allows the mental access is the totality of all *attributes* that belong to all those things, and only those things, that should be grasped by that 'concept'. In his commentary to the Aristotelian *'Peri hermeneias'* (Περὶ ἑρμηνείας) he referred to the human *'intellect'* as that part of the mind that is able to perform the above mentioned collections of attributes:

> *'The things are comprehended by the intellect, ...'* (*"Res enim ab intellectu concipitur, ...",* cf. Boëthii Commentarii in librum Aristotelis Περὶ ἑρμηνείας, recensuit Carolus Meiser, Pars posterior, Leipzig 1880, p. 21).

Here the reference to the *'intellect'* is quite important. In fact, the Latin word *'intellectus'* is derived from the verb *'intellegere'* (inter + legere) which means basically *'to find out all attributes that can be attributed to a thing, and to collect them all'.* Therefore, the *'intellect'*

is understood as that part of the soul (or the mind), in which sense-perceptions are critically examined and thereby transformed into cognitions. It is, hence, the *'intellect'* which is able to recognize those attributes that are common to all objects of a given class of things (see e.g. CICERO *'Ad Marcum Brutum Orator'*, §33, or CICERO *'Topica'*, §5, in which the original meaning of the term *'intellectus'* is witnessed). Notice that the Boëthian term *'conceptio animi'* was changed during the Middle Ages into *'conceptus'* (e.g. in the works of PETRUS ABAELARDUS, WILLIAM OF OCCAM and others).

It is also worth noting that BOËTHIUS' doctrine of concepts is not limited to the creation of *representations* of what the senses have brought to the consciousness of the soul. Rather it is required that what the soul has pondered should be *expressible* in a familiar language. That what a thing is ('quid sit res'), its *quidditas*, must be expressible in words. The *'concept'* (*conceptio animi, conceptus*) of a certain thing, or more general of a species or genus, is, hence, the collection of all those attributes which belong to the quidditas of the thing, resp. to the quidditas of each thing that belongs to the species, or genus under consideration.

The contributions of BOËTHIUS' to the doctrine of concepts were highly estimated during the Middle Ages. However, almost all of his works were almost forgotten during the Renaissance and the following centuries. However his contribution to the doctrine of *'concepts'* was still remembered by quite a number of philosophers. In the seventeenth century his contributions were repeated and put into new light in the highly influential treatise *'La Logique ou L'Art de Penser'* (Paris 1662) by ANTOINE ARNAULD and PIERRE NICOLE, also called *'Logique de Port-Royal'*. There we find in Part I, Chapter 7, the following precise definition of the classical understanding of the notion of 'concept':

> *"J'appelle compréhension de l'idée, les attributs qu'elle enferme en soi, et qu'on ne lui peut ôter sans la détruire, ..."*
> [I call *'content of an idea'* (the totality of) all attributes that the idea encloses in itself and that cannot be removed from it without destroying the idea.]

What in this citation is called *'content of an idea'* is that what was called *'conceptio animi'* by BOËTHIUS and is now called simply a *'concept'*. This, however, is the classical understanding of concepts and refers to classes of attributes written down in an everyday language.

However, at the end of the period called Renaissance, it turned out that this classical understanding of *'concepts'* was no longer adequate in Natural Sciences and in Mathematics and probably also in other sciences. The reason is that for BOËTHIUS and ARNAULD & NICOLE *concepts* are built only in situations where *attributes* play a significant role and the descriptions of the concepts are given in everyday languages and thus, in general, cannot be considered as defined sufficiently precise.

Introducing concepts using measurement methods. In the modern Natural Sciences which emerged in the early 17th century and in which it was aimed at discovering general laws, which govern physical processes, the classical understanding of concepts was too narrow. The notions of *'force'*, of *'impulse'*, of *'mass'*, of *'temperature'* etc., are examples of phenomena which are not 'things' and hence cannot be characterized by 'attributes'. For

example, the effort, that is called *'force'* in everyday language is nothing more than an immediate sensation which has no precise definition. Turning away from everyday language a slightly different concept of *'force'* was introduced into Physics, not by means of attributes but by means of mathematical equations: the size of *force* exerted by an object is the arithmetical product of the size of its *mass* and the size of its *acceleration*. In this definition, however, it is not defined what *'force'* essentially is, but only the way to measure its size. This opened a new way of introducing basic concepts rigorously into Natural Sciences (GALILEO GALILEI, ISAAK NEWTON, et al.).

Introducing concepts using implicit definitions. In Mathematics, as we have seen, the problem of precisely defining its basic *concepts* also emerged in the early 17th century (DESCARTES, PASCAL, TSCHIRNHAUS, et al.). A first indication of how this could be achieved was given by JOHANN HEINRICH LAMBERT (1728-1777) in 1766. In his investigations concerning the Problem of Parallel Lines he suggested that the basic terms of geometry, which appear in the list of Euclidean postulates and, although they are formulated in colloquial language, should be treated as *'empty terms'.* This treatment should guarantee that all those contents are eliminated that might be hidden in the terms but are undesired. Only those relations between the basic terms that are expressed in the postulates would then remain.

LAMBERT's suggestion was rather bold at his time, but impressed quite some mathematicians and logicians in the following decades. JOSEF DIEZ GERGONNE (1771-1859) in his *'Essay sur la Théorie des Definitions'* (Ann. de Math. 9 (1818), pp. 1-35) proposed to call LAMBERT's suggested procedure an attempt to define the basic concepts *'implicitly'.* LAMBERT's suggestion was fundamental in MORITZ PASCH's *'Vorlesungen über neuere Geometrie'* (Leipzig, 1882) and similarly in G. PEANO's *'I Principii Geometria Logicamente Esposti'* (Turino 1889) and also most fundamental in DAVID HILBERT's *'Grundlagen der Geometrie'*, Leipzig 1899.

In all of these expositions of the foundations of geometry LAMBERT's suggestion to define the basic concepts *implicitly* was of crucial importance. These definitions are called *'implicit definitions'* because the (empty) signs are woven or 'folded' (*plicare* (lat.) = to fold) into the system of axioms, and that system as a whole *limits* the possibilities of interpreting these empty signs. Although the introduction of *'implicit definitions'* was attacked very heavily by numerous philosophers and mathematicians at that time (e.g. by GOTTLOB FREGE, see his *'Kleine Schriften',* Hildesheim 1967, p. 268ff, p. 286ff, and others), it should be clear that *'implicit definitions'* are the only possibilities of introducing the basic concepts in a mathematical theory. This kind of defining concepts goes far beyond the classical way of introducing concepts that we mentioned at the beginning.

E.2 Mathematical theories are defined by the *'frameworks of their concepts'*

It was DAVID HILBERT, who, in 1899, called an axiom system in which the basic concepts are introduced as empty signs, but obtain their meaning and significance by being put into relation with other basic concepts, a *"framework of concepts"*. In fact, the nature (or substance) of the objects mathematics is dealing with is not at all important. Only the relations between the objects are of importance. This has been emphasized by many

mathematicians over and over again and we reported on this in Chapter 19.

We remember that DESCARTES, in his *'Discours de la Méthode'* (1637), had already pointed out that, in mathematics, one does not study the nature of individual objects but

> *"merely the various relationships and proportions"*

that they mutually have (DESCARTES, Œuvres VI, p. 20). POINCARÉ had added, in his book *'La Science et l'hypothèse'* (1902) (we talked about this in Chapter 19), that

> *'mathematicians do not study the objects, but the relations between the objects, and that it is indifferent to them whether these objects are replaced by others or not, if only the relations remain unchanged.'*

So, when constructing a mathematical theory, nothing needs to be said about the nature (or matter) of the objects that the theory is dealing with. A mathematical theory is, as DAVID HILBERT described it in his lecture *"Axiomatisches Denken"* (*'Axiomatic Thinking',* Zürich, 1917), nothing more than a *'framework of concepts'* (cf. Chapter 20). The point here is that *not the objects* of a mathematical theory are defined *but only the theory* (cf. also our commentary to HANKEL's theory of negative numbers in Chapter 18, p. 299, lines 1-7).

So, when constructing a mathematical theory, it is only necessary to say that the theory has to do with some *'systems of things'* and that the things contained in these systems are *'in certain mutual relations'. - "Die genaue und für mathematische Zwecke vollständige Beschreibung dieser Beziehungen erfolgt durch die Axiome"* dieser Theorie *('The exact and, for mathematical purposes, complete description of these relations is given by the axioms'* of this theory).

The reader will have noticed that, in the previous paragraph, we have taken all of the quotations used from HILBERT*'s* book on the *'Grundlagen der Geometrie'* (1899) and from the 'Declaration' (cf. Ch. 20), which precedes it. The insights to which the previous chapters have led us can now be summarized as follows:

A mathematical theory is completely outlined as soon as all of its axioms are written down in a suitable formal language $\mathscr{L}$. In $\mathscr{L}$, all basic concepts are denoted by empty signs. The axioms describe the relationships between the objects in question and all that what is assumed to exist. In this way, the empty signs receive all of those contents that are mathematically relevant.

We emphasize once again that, in order to construct a mathematical theory, the realm of objects that are to be treated in the theory does not need to be given in advance. Such a realm of objects would not usually be found anyway, neither in the sense-perceivable world nor in any supernatural world.

In order not to be misunderstood, we also emphasize that, since antiquity, in a mathematical theory, although it will usually be concerned with objects of the real world, these objects are not investigated as they are, but only in suitable abstract and idealized forms. Strictly speaking, these abstract and idealized objects are not identical to the objects of the real world, and, moreover, cannot be found among the real existing objects. (The empty signs, e.g., appear in the formulation of the framework of concepts in order that we

may have complete control over the amount of abstraction and idealization underlying the formation of the theory, cf. Chapter 19.)

The amount of idealization and abstraction cannot prevent that the resulting theory perhaps might be inconsistent. Thus, the problem of consistency appears. If the theory is free of contradictions, then it will be possible to construct a domain of objects on the basis of a formal theory of sets, e.g., ZSF. It would be a model of the theory, but cannot be constructed until the axiom system for the theory is fully worked out. Thus, again, the starting point for the drawing up of a mathematical theory is its framework of concepts.

E.3 The Objects of a Mathematical Theory

In principle, it would be quite sufficient to base our mathematical work exclusively (!) on the framework of concepts underlying the mathematical theory we are working with, on all its formalisms, calculi and axiom systems. No undesired ontological or ideological assumptions would then be incorporated into the theory. The only assumption upon which we have to go is that the theory is free of contradictions. However, according to GÖDEL's theorems (cf. Chapter 20), the proofs of freedom from contradiction required for this are only feasible in rather simple cases.

However, every single proof presented can be checked mechanically (in principle, at least) as to whether it has been carried out correctly on the basis of the completely formalized underlying language, the logical derivation rules and the axioms of the theory. A general method for making mathematical thinking safe and secure is not available (and will not be available, according to GÖDEL's theorems). But it is possible to safeguard and secure each individual proof.

But no one is willing to give up any reference to mathematical objects.

For many reasons, it is desirable to have objects that are denoted by the signs of the formal axiom systems, so as not to have to restrict all mathematical work to dealing with series of signs of formal systems (i.e., the syntax of formal systems), but to be able to deal with mathematical objects in thought.

In earlier times, people liked to say that the signs of a mathematical theory either denote real, natural objects or just purely *fictitious* objects, i.e., objects 'created' by the mind. But is the human mind able to create things? The question here is whether the human mind is able to create such things that do have certain properties *per se* (καθ' αὑτό), i.e., as *attributes that essentially belong to them*. This question has never found a convincing positive answer, which means that the answer to the question is, obviously, in the negative.

But, on the basis of a sufficiently strong formal axiomatic theory of sets, the 'existence' of the objects in question can be achieved in a certain sense, namely, as elements of certain *'structures'*, i.e., as elements of *models* of the theories under consideration (cf. Chapter 19). In these *'structures'*, the elements do not necessarily have the desired properties *'per se',* but they do have them in the sense of the model, i.e., *'per accidens'* (κατὰ συμβεβηκός).

This is a highly remarkable answer to the main question that we posed in the introduction to this book, namely, in what sense the objects of the different mathematical

theories exist. We have seen that these objects cannot be found as real things that have the desired properties as attributes. But the structuralism opened the possibility for a convincing new answer*: these objects may exist as elements of suitable 'structured sets', i.e., as elements of suitable 'structures', and these elements possess the desired properties in the sense of the structure, that is, 'per accidens'.*

E.4 Deepening the level of the foundations

The transformation of mathematics that began in the late 19th century was described by many as a *paradigm shift.* Some even spoke of *'Modern Mathematics'.* They meant that, now, in the 'modern' era, mathematicians had broken with some important and longstanding traditions. The transformation of mathematics was indeed enormous. However, it did not break with its previous development, but essentially only deepened its foundations. This deepening is connected to the terms *axiomatics, set theory, logic* and *structuralism* (cf. Chapters 15, 19 and 20). It has changed the appearance of mathematics - much to its advantage.

We can now answer the two questions that were posed at the beginning:

The sources from which we can draw when we want to prove mathematical theorems are no longer hidden: these sources are the axioms of the respective theories. In the axioms of a theory, the entire content of the theory is contained, like in a nutshell (cf. Chapter 4).

The system of axioms is nothing more than a system of propositional forms in which all the basic concepts of the theory are implicitly defined. The axioms are, hence, nothing else than (implicit) 'definitions'. It is, hence, also the case that in the (implicit) definitions of a theory the entire content of the theory is contained, like in a nutshell.

The mathematical objects exist only insofar as they appear as elements of suitable structures and will have the required properties in the sense of these structures, i.e., per accidens (κατὰ συμβεβηκός). *They do not have any other kind of existence.*

The extensive field of the philosophy of mathematics has only been covered here in this book in a few sections. We had to be selective. We have already explained the selection we made in the preface and in the introduction: it is in aid of the search for answers to the two questions mentioned above.

With the presentation of HILBERT's axiomatic treatment of mathematical theories (Chapter 20) and the related structuralist position (Chapter 19) based on set theory (Chapter 15) and mathematical logic, we have advanced to a well-founded answer to our questions. We hope that we have also convinced our readers with this fundamental understanding of mathematics as a whole.

AD LECTOREM. *Si plura desideras, amice lector, consule ipsum auctorem. Vale.*

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Index of Abbreviations

ca. (=circum ea):	circa
cf. (= conferatur):	compare, it is contributed for comparison
e.g (= exempli gratia):	for example
i.e (= id est):	that is to say
loc. cit. (= loco citato):	at the specified location
op. cit. = (opus citatum):	the work cited in the list of references
p. (= pagina):	page
pp. (= paginae):	pages
B.C.E. (= before common era):	Indication of a year that precedes today's usual chronology
DPA (= DEDEKIND-PEANO Arithmetic):	i.e. the first-order theory of Arithmetic based of the axioms of Dedekind and Peano
ZSF (= ZERMELO-SKOLEM-FRAENKEL Set Theory):	i.e. the first-order theory of sets based on the axioms of Zermelo, Skolem and Fraenkel, sometimes also called ZF.

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